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FORCE AND MOMENTS ON ASYMMETRIC AND YAWED BODIES IN A FREE SURFACE

Nabil Abdel-Hamid Daoud

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Introduction

The potential flow around asymmetrical bodies in the presence of a free surface is of interest in many problems of ship hydrodynamics. Asymmetry in ships can either be permanent as in catamarans, or temporary due to rotations of symmetric hulls in yaw or roll. The purpose of this work is to investigate theoretically the forces and moments acting on such bodies for the case of a steady translational motion in an ideal fluid. The solutions are limited to thin bodies with a small degree of asymmetry. These restrictions are employed so as to yield a well defined boundary-value problem having a solution which lies within the scope of the potential theory.

The treatment of hydrodynamic problem was based mainly on information deduced from the theory of low-aspect ratio wings. Davidson and Schiff (1946) pointed out that larger changes in the wave-making pattern are observed when the Froude number is larger than 0.19. Hu (1961) solved this problem for a yawed ship by the use of an iterative perturbation method based on the asymptotic expansion of the Kernel function of the integral equation for small Froude numbers. He found that the magnitudes of the forces and moments acting on the ship increase rapidly as the Froude number increases up to $F_n = 0.35$ and then remain mainly constant.

The solution of the boundary-value problem leads to a Fredholm integral equation of the first kind for the doublet moment distribution similar to the usual representation of a lifting surface. Due to the complicated form of the Kernel function approximate numerical methods must be applied to solve this integral equation.

In this work a numerical solution is used based on the finite-element method, where the region of integration is divided into small rectangular elements and the unknown doublet moments are defined by an approximating function within each element. In doing that and performing the integrations within each element analytically, the integral equation reduces to a set of linear algebraic equations that can be solved for the unknown doublet strength. We carry through this procedure for two cases, one when $L/T=20$ and one when $L/T=7$. The results are summarized in the concluding section of this work.

I. Mathematical Formulations

It will be convenient in formulating this problem to introduce three right-handed coordinate systems. One is fixed in space, $\overline{Oxyz}$, with $\overline{Oy}$ directed opposite-ly to the force of gravity, $\overline{Ox}$ coincides with the direction of motion, and $\overline{Oxz}$ lies in the plane of the undisturbed free surface. Of the other two coordinate systems, one is fixed in the body, $O'x'y'z'$, and one is moving with the body, Oxyz, but is taken in such a way that the (x,z)-plane coincides with the mean water surface, and Ox makes an angle δ with Ox. Further, Oxyz and $O'x'y'z'$ coincide when the body is at rest (see Figure 1).

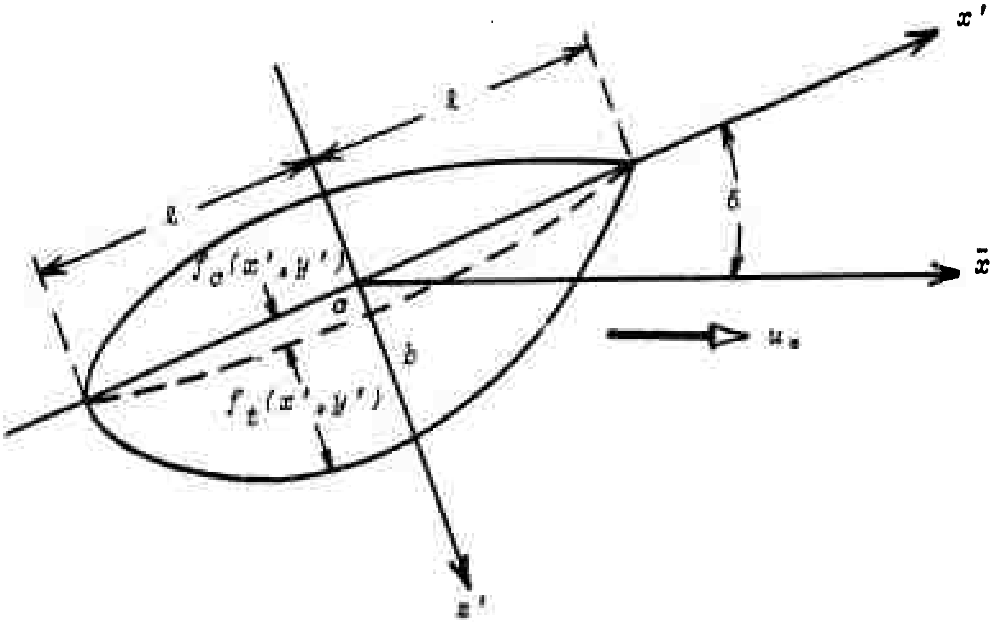


Figure 1

Plan View of an Asymmetric and Yawed Body

The hull form will be described by the surface

$H_1(x',y',z') = 0$
or more explicitly by

$z' = f_s(x',y')$ for the starboard surface

and $z' = f_p(x',y')$ for the port surface.

Define $f_c(x',y')$ and $f_t(x',y')$ by

$f_c(x',y') = 1/2\,[f_s(x',y') + f_p(x',y')]$

$f_t(x',y') = 1/2\,[f_s(x',y') - f_p(x',y')]$

The functions f_c and f_t will be assumed to have continuous derivatives with respect to each variable, $z' = f_c(x',y')$ is the position of the "mean camber surface," and $2f_t(x',y')$ is the local thickness. Thus the equation of the hull surface can be written in the form

$$H_1(x',y',z') = z' - [f_c(x',y') \pm f_t(x',y')] = 0 \tag{I-1}$$

When the body is moving, its trim, heel and the position of its center of gravity relative to Oxyz will change. Let α be the trim angle, measured positively in the bow-up direction, θ be the heel angle measured positively clockwise and let h be the amount by which the origin $0'$ is raised (see Figure 2). Then Oxyz and $O'x'y'z'$ are related by the equations

$$x' = x\cos\alpha + [(y - h)\cos\theta + z\sin\theta]\sin\alpha$$

$$y' = -x\sin\alpha + [(y - h)\cos\theta + z\sin\theta]\cos\alpha$$

$$z' = -(y - h)\sin\theta + z\cos\theta$$

and $$x = x' \cos\alpha - y' \sin\alpha \tag{I-2}$$

$$y = h + [x' \sin\alpha + y' \cos\alpha] \cos\theta - z' \sin\theta$$

$$z = [x' \sin\alpha + y' \cos\alpha] \sin\theta + z' \cos\theta$$

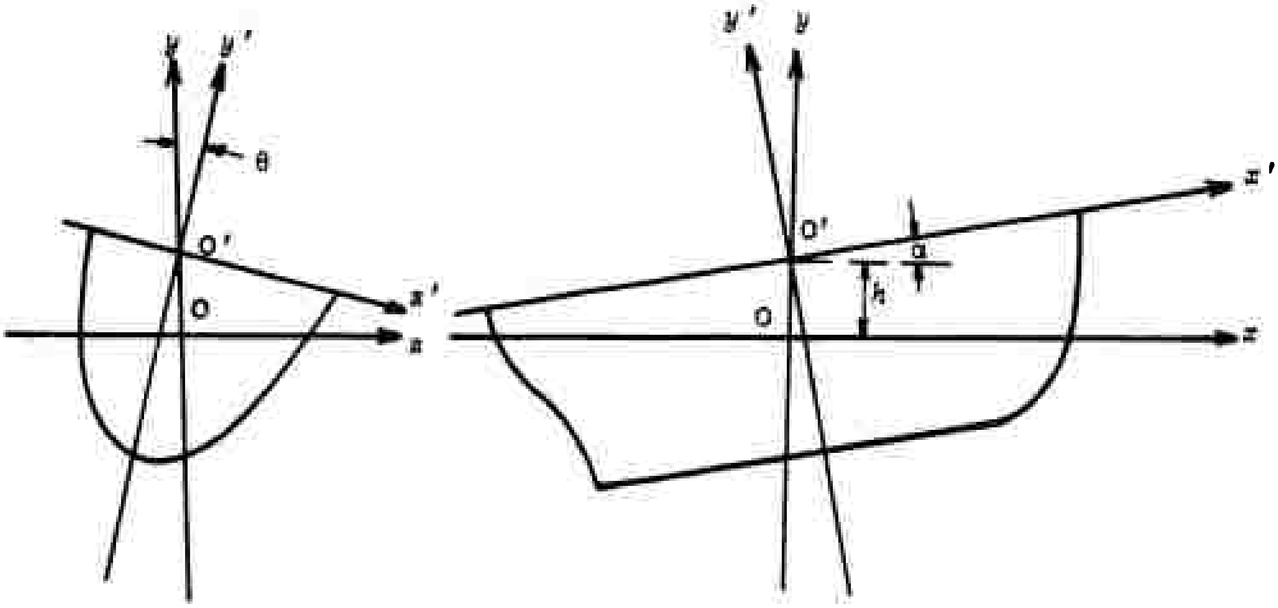


Figure 2

Coordinate Axis

Let us define

$$H(x,y,z) = -(y - h) \sin\theta + z \cos\theta - [f_c(x'(x,y,z),\ y'(x,y,z))$$

$$\pm f_t(x'(x,y,z),\ y'(x,y,z))] \tag{I-3}$$

which describes the hull in the system Oxyz.

The motion of the fluid, which is assumed irrotational, is most easily described in the systems $\overline{Oxyz}$

or Oxyz. Let $\Phi(\bar{x},\bar{y},\bar{z})$ be the velocity potential in the fixed system and $\phi(x,y,z)$ that in the moving system. Then

$$\Phi(\bar{x},\bar{y},\bar{z}) = \phi(\bar{x} - u_o \cos\delta\cdot t,\ \bar{y},\ \bar{z} - u_o \sin\delta\cdot t) \qquad \text{(I-4)}$$

Both the velocity potentials Φ and ϕ satisfy Laplace's equation

$$\Delta\Phi \equiv \Phi_{\bar{x}\bar{x}} + \Phi_{\bar{y}\bar{y}} + \Phi_{\bar{z}\bar{z}} = 0$$
$$\Delta\phi \equiv \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad \text{(I-5)}$$

The absolute velocity of the water described in the moving coordinates Oxyz has the components

$$(u - u_o \cos\delta,\ v,\ w - u_o \sin\delta) = (\phi_x - u_o \cos\delta,\ \phi_y,\ \phi_z - u_o \sin\delta)$$

This velocity is continuous everywhere in the fluid except possibly at the wake, which will be assumed to occupy a thin sheet that joins the body at its aft end and extends infinitely along the negative x-axis. For more discussions concerning this assumption we refer to Thwaites (1960) or Robinson & Laurmann (1956, 1.15). The pressure in the fluid can be computed from Bernoulli's integral, once ϕ is found, as

$$P/\rho = u_o(\phi_x \cos\delta + \phi_z \sin\delta) - gy - 1/2(\phi_x^2 + \phi_y^2 + \phi_z^2) \ . \qquad \text{(I-6)}$$

Let the equation of the free surface be

$$y = Y(x,z) = Y(\bar{x} - u_o \cos\delta,\ \bar{z} - u_o \sin\delta) \ . \qquad \text{(I-7)}$$

Then the potential function $\phi(x,y,z)$ must satisfy two boundary conditions on this surface. The first is the kinematic boundary condition

$$\phi_x\big(x,\ Y(x,z),\ z\big)\ Y_x(x,z) - \phi_y + \phi_z\ Y_z =$$
$$= u_o(Y_x \cos\delta + Y_z \sin\delta) \ . \qquad \text{(I-8)}$$

The second is the dynamical boundary condition

$$gY(x,z) - u_o[\phi_x(x,\ Y(x,z),\ z)\ \cos\delta + \phi_z \sin\delta] + \\ + 1/2[\phi_x^2 + \phi_y^2 + \phi_z^2] = 0 \quad . \qquad \text{(I-9)}$$

There are corresponding kinematic and dynamic boundary conditions to be satisfied on the body's wetted surface S_w . The kinematic condition is

$$-(u - u_o\cos\delta)\ H_x(x,y,z) - vH_y(x,y,z) - (w - u_o\sin\delta)\ H_z(x,y,z) = 0$$

$$\phi_x|_{S_w}\ H_x(x,y,z) + \phi_y|_{S_w}\ H_y(x,y,z) + \phi_z|_{S_w}\ H_z(x,y,z) =$$

$$= u_o[H_x(x,y,z)\ \cos\delta + H_z(x,y,z)\ \sin\delta] \quad , \qquad \text{(I-10)}$$

where

$$H_x(x,y,z) = -(f_{cx'} \pm f_{tx'})\ \cos\alpha + (f_{cy'} \pm f_{ty'})\ \sin\alpha \quad ,$$
$$H_y(x,y,z) = -\sin\theta - [(f_{cx'} \pm f_{tx'})\ \sin\alpha + (f_{cy'} \pm f_{ty'})\ \cos\alpha]\ \cos\theta \quad ,$$
$$H_z(x,y,z) = \cos\theta - [(f_{cx'} \pm f_{tx'})\ \sin\alpha + (f_{cy'} \pm f_{ty'})\ \cos\alpha]\ \sin\theta \quad . \qquad \text{(I-11)}$$

The dynamical boundary conditions on the body are simply the equations of static equilibrium of the forces acting on the body. These equations can be written down in various ways to conform to the physical situation of the problem. Here we will confine ourselves to finding the hydrodynamic force components acting upon the body and the moments, about the origin 0'. Denote the projection of the wetted hull onto the $O'x'y'z'$ plane by S_{wp} , the water pressure on the starboard side of the hull by $P_s(x,y,z)$ and the pressure on the port side of the hull by $P_p(x,y,z)$. Then

$$\vec{F} = \iint_{S_w} P(x,y,z)\ \vec{n}\ ds \quad ,$$

$$F_{x'} = \iint_{S_{wp}} [P_s(x,y,z)\, f_{sx'} - P_p(x,y,z)\, f_{px'}]\, dx'\, dy' \quad ,$$

$$F_{y'} = \iint_{S_{wp}} [P_s(x,y,z)\, f_{sy'} - P_p(x,y,z)\, f_{py'}]\, dx'\, dy' \quad ,$$

$$F_{z'} = \iint_{S_{wp}} [-P_s(x,y,z) + P_p(x,y,z)]\, dx'\, dy' \quad . \tag{I-12}$$

$$\vec{m} = \iint_{S_w} P(x,y,z) \cdot (\vec{r} \times \vec{n})\, ds \quad ,$$

$$m_x = \iint_{S_{wp}} \left\{P_s \cdot (-y' - z'f_{sy'}) - P_p(-y' + z'f_{py'})\right\} dx'\, dy' \quad ,$$

$$m_y = \iint_{S_{wp}} \left\{P_s \cdot (z' \cdot f_{sx'} + x') - P_p \cdot (-z'f_{px'} + x')\right\} dx'\, dy' \quad ,$$

$$m_z = \iint_{S_{wp}} \left\{P_s \cdot (x'f_{sy'} - y'f_{sx'}) - P_p \cdot (x'f_{py'} - y'f_{px'})\right\} dx'\, dy' \quad . \tag{I-13}$$

A kinematic condition must also be satisfied on the ocean bottom. Where this last is assumed to be of infinite depth, the condition may be written as

$$\lim_{y\to\infty} \phi_y = 0 \quad . \tag{I-14}$$

Finally there are the conditions at infinity, to insure that waves will only follow the body

$$\phi(x,y,z) = \begin{cases} O\left([x^2 + z^2]^{-1/2}\right) & \text{as } x^2 + z^2 \to \infty \text{ for } x > 0 \quad , \\ O(1) & \text{as } x^2 + z^2 \to \infty \text{ for } x < 0 \quad . \end{cases} \tag{I-15}$$

Various modifications of this problem are possible, depending upon the physical situation. In particular, the wave resistance of a catamaran can be determined if we add one further condition analogous to (I-14) to be satisfied on its plane of symmetry.

II. Method of Solution

One of the properties of the problem formulated in the preceding chapter which makes it mathematically intractable is that it is nonlinear. In order to obtain a solution, we will use the mothod of perturbation expansion as an approximation to linearize the problem.

Perturbation Expansion:

It is obvious from the nature of the problem under consideration that the disturbance caused near the free surface is dependent on two perturbation parameters. One represents the "thinness" effect of the body β , where in the limit as $\beta \to 0$, the body degenerates to a cambered plane of zero thickness. The other parameter ε describes the asymmetrical effects due to camber and incidence angle. As $\varepsilon \to 0$, the problem becomes one of a symmetrical body aligned with the incident flow.

We begin by imbedding the hull form (I-1) in a family of hulls as follows

$$z' = \varepsilon f_c^{(1)}(x',y') \pm \beta f_t^{(1)}(x',y') \quad . \tag{II-1}$$

Also δ can be written in the form

$$\delta = \varepsilon\delta^{(1)} \ . \tag{II-2}$$

We tentatively assume that all the physical variables can now be expanded as an asymptotic series in terms of the two parameters β and ε . Thus we obtain as the basic expansions

$$\begin{aligned}
\phi(x,y,z;\beta,\varepsilon) &= \beta\phi^{(10)}(x,y,z) + \varepsilon\phi^{(01)}(x.y,z) + \beta\varepsilon\phi^{(11)} +\ldots, \\
Y(x,z;\beta,\varepsilon) &= \beta Y^{(10)}(x,z) + \varepsilon Y^{(01)}(x.z) + \beta\varepsilon Y^{(11)} +\ldots, \\
\alpha(\beta,\varepsilon) &= \beta\alpha^{(10)} + \beta\varepsilon\alpha^{(11)} +\ldots, \\
h(\beta,\varepsilon) &= \beta h^{(10)} + \beta\varepsilon h^{(11)} +\ldots, \\
\theta(\beta,\varepsilon) &= \beta\theta^{(10)} + \beta\varepsilon\theta^{(11)} +\ldots \ .
\end{aligned} \tag{II-3}$$

Here it should be noted that this analysis may not be applicable where the solution of the problem is singular, such as near the leading edge of the body. We refer to Van Dyke (1964, 4.4) for more discussion concerning the treatment of such problems.

The problem may now be linearized by substituting the foregoing expansions in Laplace's equation and the boundary conditions, besides expanding in a Taylor series where necessary, and collecting terms of the same order. The result of these operations is a sequence of linear boundary-value problems for the potential functions $\phi^{(10)}$, $\phi^{(01)}$, $\phi^{(11)}$, ... The

first of these, which represents the differential equations for the potential function $\phi^{(01)}$, is the well-known problem for the wave resistance of thin ships. The velocity potential $\phi^{(01)}$, which is the first-order term due to the asymmetry of the body, must satisfy the following equations :

$$\Delta\phi^{(01)} = 0 \ , \qquad\qquad y < 0 \ ,$$

$$\phi^{(01)}_{xx}(x,0,z) + \kappa\phi^{(01)}_{y} = 0 \ , \qquad\qquad \kappa = g/u_o^2 \ ,$$

$$\phi^{(01)}_{z}(x,y,\pm 0) = u_o(\delta^{(1)} - f^{(1)}_{cx}) \quad on \quad S_o \ ,$$

$$\lim_{y\to-\infty} \phi^{(01)}_{y}(x,y,z) = 0 \ ,$$

$$\phi^{(01)} = \begin{cases} O\left([x^2 + z^2]^{-1/2}\right) \\ O(1) \end{cases} \quad as \quad x^2 + z^2 \to \infty \quad for \quad \begin{cases} x > 0 \ , \\ x < 0 \ . \end{cases} \tag{II-4}$$

The free surface $Y^{(01)}$ is determined by

$$Y^{(01)}(x,z) = u_o/g\phi^{(01)}_{x}(x,0,z) \ . \tag{II-5}$$

The velocity potential $\phi^{(11)}$ represents the lowest-order term due to the combined effect of thickness and camber and/or the incidence angle. The solution for this potential function will not be given here, though it may be of value for investigations in the

future. We will confine ourselves in the subsequent analysis to finding the velocity potential $\phi^{(01)}$. First let us apply the method of Green functions to solve the boundary-value problem (II-4) for this velocity potential.

Method of Green Functions:

As is usual in this method, it relies upon the ability to construct a function of the form

$$G(x,y,z;\xi,\eta,\zeta)=G(P;Q)=r^{-1}+G_o(x,y,z;\xi,\eta,\zeta),$$

where

$$r=\left((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right)^{1/2}$$

and G_o is harmonic in the region occupied by fluid, or in the case of this linearized problem, in the region below the equilibrium free surface.

Dimensionless quantities will be used in all the expressions hereafter, by comparing lengths to half the length of the body ℓ and velocities to u_o . Also,

$$\phi(x,y,z)=u_o\ell\tilde{\phi}(\tilde{x},\tilde{y},\tilde{z}), \qquad G=1/\ell\tilde{G} .$$

Equations (II-4) will then become, after dropping the tildes,

$$\Delta\phi^{(01)}=0, \qquad y<0,$$

$$\phi_{xx}^{(01)}(x,0,z)+\gamma_o\phi_y^{(01)}=0, \qquad \gamma_o=g\ell/u_o^2,$$

$$\phi_z^{(01)}(x,y,\pm 0) = \delta^{(1)} - f_{\alpha x}^{(1)} \quad \text{on } S_o \ ,$$

$$\lim_{y\to-\infty} \phi_y^{(01)}(x,y,z) = 0 \ ,$$

$$\phi^{(01)} = \begin{cases} O([x^2 + z^2]^{-1/2}) & \\ & \text{as } x^2+z^2 \to \infty \text{ for } \begin{cases} x > 0 \ , \\ \\ x < 0 \ . \end{cases} \\ O(1) & \end{cases}$$

(II-6)

where S_o is that part of the xy plane which is bounded by $-1 < x < 1$ and $-\tau < y < 0$, and τ is the dimensionless draft.

We require the following of G :

$$\Delta G = 0 \ ,$$

$$G_{\xi\xi}(x,y,z;\xi,0,\eta) + \gamma_o G_\eta = 0 \ ,$$

$$\lim_{\eta\to-\infty} G_\eta = 0 \ ,$$

$$G = \begin{cases} O([\xi^2 + \zeta^2]^{-1/2}) & \\ & \text{as } \xi^2+\zeta^2 \to \infty \text{ for } \begin{cases} \xi < 0 \ , \\ \\ \xi > 0 \ , \end{cases} \\ O(1) & \end{cases}$$

(II-7)

Consider now the region of fluid bounded by the free-surface plane Σ_F , the two sides of S_o and the wake w , a circular cylinder Σ_R with Oy as the axis and a radius R , and a horizontal plane at $y = -\infty$ closing the bottom of the cylinder. Then by Green's Theorem we have the following formula

$$4\pi\phi^{(01)}(P) = \iint\limits_{S(Q)} \left(\phi_\nu^{(01)}(\xi,\eta,\zeta)\, G(P;\xi,\eta,\zeta) - \phi^{(01)} G_\nu\right) ds \ ,$$

$$= \iint\limits_{S_o+\omega} \Big\{ \Big(\phi_\zeta^{(01)}(\xi,\eta,+0)\, G(P;\xi,\eta,0) -$$

$$- \phi^{(01)}(\xi,\eta,+0)\, G_\zeta(P;\xi,\eta,0)\Big) -$$

$$- \Big(\phi_\zeta^{(01)}(\xi,\eta,-0)\, G(P;\xi,\eta,0) -$$

$$- \phi^{(01)}(\xi,\eta,-0)\, G_\zeta(P;\xi,\eta,0)\Big)\Big\}$$

$$+ \iint\limits_{\Sigma_F} \left(\phi_\eta^{(01)}(\xi,0,\zeta)\, G(P;\xi,0,\zeta) - \phi^{(01)} G_\eta\right) d\xi d\eta$$

$$- \lim_{\eta\to-\infty} \iint\limits_{\Sigma_B} \left(\phi_\eta^{(01)}(\xi,\eta,\zeta)\, G(P;\xi,\eta,\zeta) - \phi^{(01)} G_\eta\right) d\xi d\eta$$

$$+ \int_0^{2\pi} d\theta \int_{-\infty}^0 d\eta\, R(\phi_R^{(01)} G - \phi^{(01)} G_R) \ .$$

Substituting (II-6) and (II-7) in the above formula, we obtain

$$4\pi\phi^{(01)}(P) = - \iint\limits_{S_o+\omega} G_\zeta(P;\xi,\eta,0)\, [\phi^{(01)}(\xi,\eta,+0) - \phi^{(01)}(\xi,\eta,-0)]\, d\xi d\eta$$

$$- \frac{1}{\gamma_o} \iint\limits_{\Sigma_F} \frac{\partial}{\partial\xi} [\phi_\xi^{(01)} G - \phi^{(01)} G_\xi]\, d\xi d\zeta$$

$$+ \int_0^{2\pi} d\theta \int_{-h}^0 d\eta\, O(1) \ .$$

The second integral may be integrated by parts to give

$$- \frac{1}{\gamma_o} \oint [\phi_\xi G - \phi G_\xi]\, n_1\, ds \ ,$$

where this line integral is taken around the intersection

of Σ_F and Σ_R . As $R\to\infty$, both this integral and the third one converge to zero. We finally have

$$\phi^{(01)}(x,y,z)=\frac{-1}{2\pi}\iint_{S_o+w} G_\zeta(P;\xi,\eta,0).$$

$$\left(\phi^{(01)}(\xi,\eta,+0)-\phi^{(01)}(\xi,\eta,-0)\right)d\xi d\eta$$

(II-8)

It should be noted here that the above analysis was done in more detail following Wehausen (1963) and the same final result was obtained.

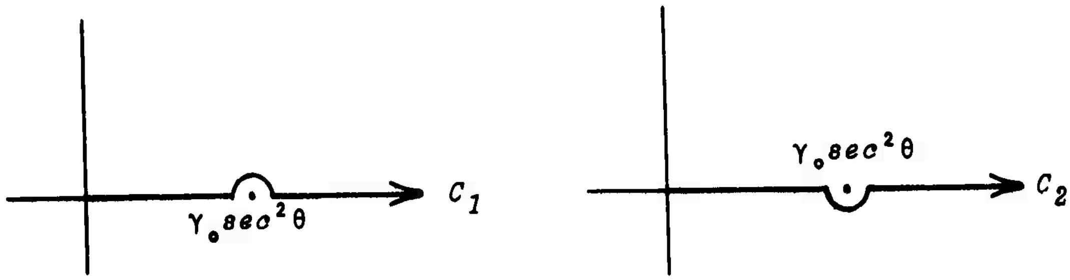
The solution for the Green functions is well known for several physical situations, many of which are given in Wehausen and Laitone (1960). Here we use one form given by Eggers and others (1960), which is suitable in our analysis.

$$G(x,y,z;\xi,\eta,\zeta)=\frac{1}{r}-\frac{1}{r_1}-Re\frac{\gamma_o}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sec^2\theta d\theta .$$

$$X\left\{\int_{c_1}\frac{e^{\kappa\left((y+\eta)+i\bar{\omega}_1\right)}}{\kappa-\gamma_o sec^2\theta}dk+\int_{c_2}\frac{e^{\kappa\left((y+\eta)-i\bar{\omega}_1\right)}}{\kappa-\gamma_o sec^2\theta}dk\right\},$$

(II-9)

where c_1 and c_2 represent the indented paths of the contour integration around the pole as shown in the figure below



and

$$r_1=\left((x-\xi)^2+(y+\eta)^2+(z-\zeta)^2\right)^{1/2},$$

$$\tilde{\omega}_1=(x-\xi)\cos\theta+(z-\zeta)\sin\theta.$$

In handling the above countour integrals, we should be cautious since its integrands are not only oscillatory, but also possess a singularity at $\kappa=\gamma_o \sec^2\theta$ for each value of θ . It turns out, however (see Appendix A), that by a proper choice of integration path in the complex plane, this integral can be simplified considerably, namely

$$G(x,y,z;\xi,\eta,\zeta)=\frac{1}{r}-\frac{1}{r_1}-Re\,\frac{\gamma_o}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sec^2\theta d\theta\cdot$$

$$X\left(\int_0^\infty\frac{e^{-\rho}}{\rho+\rho_1}\,d\rho+\int_0^\infty\frac{e^{-\rho}}{\rho+\rho_2}\,d\rho\right)$$

$$-4\gamma_o\int_0^{\frac{\pi}{2}}H(-\omega_1)\;e^{\gamma_o\sec^2\theta(y+\eta)}\quad\sin[\gamma_o(x-\xi)\sec\theta].$$

$$\cos[\gamma_o(z-\zeta)\sin\theta\sec^2\theta]\sec^2\theta d\theta, \qquad \text{(II-10)}$$

where

$$\rho_1 = \gamma_o \sec^2\theta\,[(y+\eta)+i\bar{\omega}_1], \qquad \rho_2 = \gamma_o \sec^2\theta\,[(y+\eta)-i\bar{\omega}_1] .$$

and $H(x)$ is the Heavyside function, defined by $H(x)=1$ for $x > 0$ and $H(x)=0$ for $x < 0$.

Differentiating this function with respect to ζ, we obtain

$$G_\zeta(x,y,z;\xi,\eta,0) = \frac{z}{r^3} - \frac{z}{r_1^3} - Re\,\frac{\gamma_o^2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^4\theta\,\sin\theta\,d\theta\,\cdot$$

$$X\; i\int_0^\infty e^{-\rho}\left[\frac{1}{(\rho+\rho_1)^2} - \frac{1}{(\rho+\rho_2)^2}\right] d\rho \;-$$

$$-4\gamma_o^2\int_1^\infty H(-\bar{\omega})\;e^{\gamma_o\lambda^2(y+\eta)}\;\sin(\gamma_o\lambda(x-\xi)).$$

$$X\;\sin\left(\gamma_o\lambda\sqrt{\lambda^2-1}\cdot z\right)\lambda^2\,d\lambda$$

$$-2\gamma_o\int_1^\infty \delta(-\bar{\omega})\;e^{\gamma_o\lambda^2(y+\eta)}\;\sin[2\gamma_o\lambda(x-\xi)]\,d\lambda, \qquad \text{(II - 11)}$$

where we have made the substitution $\lambda = \sec\theta$ in the last integrals.

Determination of the Velocity Potential $\phi^{(01)}$

The integral equation for the velocity potential $\phi^{(01)}$, as in (II-8), represents a distribution of doublets over the surface S_o and the wake w. The moment $\mu(x,y)$ of such a distribution is

$$4\pi\mu(x,y) = \phi^{(01)}(x,y,+0) - \phi^{(01)}(x,y,-0). \qquad \text{(II-12)}$$

Therefore, the velocity potential $\phi^{(01)}$, after deleting the superscript $^{(01)}$, can be written in the following form

$$\phi(x,y,z) = \iint\limits_{S_o+w} -G_\zeta(x,y,z;\xi,\eta,0)\ \mu(\xi,\eta)\ d\xi d\eta \quad . \tag{II-13}$$

Now the linearized dimensionless pressure (based on the pressure $\frac{1}{2}\rho u_o^2$) associated with this velocity potential can be found from (I-6) as

$$P(x,y,z) = 2\phi_x(x,y,z)$$

Accordingly, the pressure jump across the plane $z = 0$ is

$$P(x,y,+0) - P(x,y,-0) = 2\ [\phi_x(x,y,+0) - \phi_x(x,y,-0)]\ ,$$

$$= 2\ [\phi(x,y,+0) - \phi(x,y,-0)]x \quad .$$

From (II-12), this becomes

$$P(x,y,+0) - P(x,y,-0) = 8\pi\mu_x(x,y) \tag{II-14}$$

Since the pressure must be continuous across the wake, then from (II-14), μ_x must vanish on the wake. It follows that μ is constant along lines parallel to the x-axis in the wake and therefore

$$\mu(x,y) = \mu(T,y) \quad , \qquad for\ \ x < T \quad . \tag{II-15}$$

where $x = T(y)$ is the equation of the aft end of the body. Also, since the velocity potential must be continuous everywhere outside the surface S_o and w ,

therefore it follows that

$$\mu(x,-\tau) = 0 \quad . \tag{II-16}$$

On the other hand, due to the expected singularity at the leading edge of the body, there will be a discontinuity in μ_x at this edge (see Appendix D).

Equation (II-13) may now be written as

$$\phi(x,y,z) = \iint_{W} -G_{\zeta}(x,y,z;\xi,\eta,0)\ \mu(T,y)\ d\xi d\eta \ +$$

$$+ \iint_{S_o} G_{\zeta}(x,y,z;\xi,\eta,0)\ \mu(\xi,\eta)\ d\xi d\eta \quad . \tag{II-17}$$

The velocity potential ϕ can be then determined once the distribution function $\mu(x,y)$ is known. To find μ , we apply the linearized, kinematical boundary condition (II-6) on the surface S_o to the above equation:

$$\phi_z\big|_{S_o} = \lim_{z\to 0} \frac{\partial}{\partial z} \left(\iint_{W} G_{\zeta}\mu(T,\eta)\ d\xi d\eta + \iint_{S_o} G_{\zeta}\mu(\xi,\eta)\ d\xi d\eta \right)$$

$$= -\delta + f_{cx}(x,y) \quad . \tag{II-18}$$

This is a Fredholm integral equation of the first kind for the unknown doublet moment $\mu(x,y)$. The closed analytical solution of this integral equation is beyond our resources because of the complicated nature of the kernel function. Consequently, a numerical approach, based on the finite-element method, will be used in the following chapter to obtain an approximate solution for the doublet moment $\mu(x,y)$.

III. Solution of the Integral Equation

For convenience in subsequent calculations, the surface S_o will be approximated by a rectangular plane and the coordinate system $Oxyz$ will be shifted to the system shown in Figure 3. Thus $T(y) = 0$. Let us divide S_o into rectangular elements determined by a finite number of nodal points as in Figure 3.

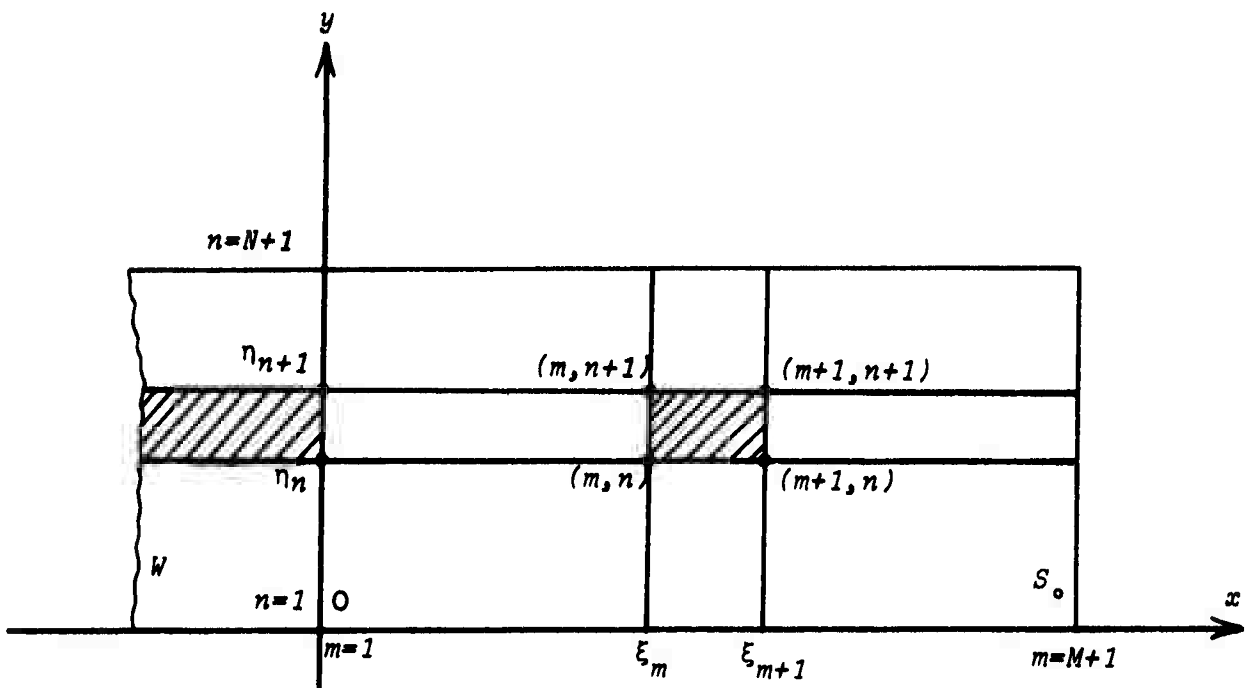


Figure 3
The Surface S_o Divided into Finite Elements

Now in order to solve the integral equation (II-18) approximately, but to any required degree of accuracy, it is convenient to assume that the presumed piece-wise

continuous function $\mu(x,y)$ over the surface S_o can be approximately determined by a finite number of values $\mu(x_m,y_n)$ at the nodal points, as described below. Where the nodal values of μ define an approximating function within each element, the integrals in (II-18) can be performed analytically over the elements and over the semi-infinite strips.

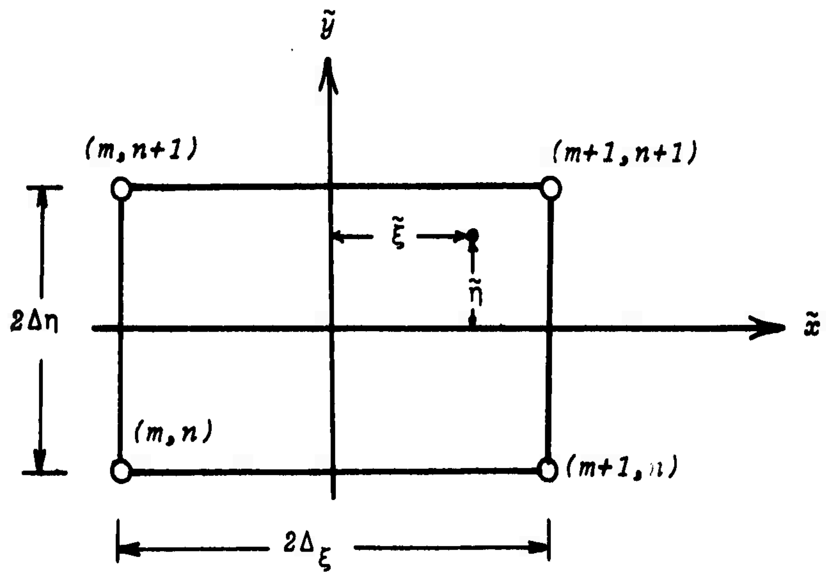


Figure 4

A Rectangular Element

Consider the rectangular element shown in Figure 4. If the function μ is to be continuous between adjacent elements it is necessary for μ to vary in a linear way along the sides of the elements. Then, coincidence of μ at the nodes will automatically insure coincidence at intermediate points. The function μ within the element may be written as

$$\mu(\tilde{\xi},\tilde{\eta}) = \frac{1}{4\Delta\xi\Delta\eta}\Big((\Delta\xi - \tilde{\xi})(\Delta\eta - \tilde{\eta})\ \mu_{m,n} +$$

$$+ (\Delta\xi + \tilde{\xi})(\Delta\eta - \tilde{\eta})\ \mu_{m+1,n} + (\Delta\xi - \tilde{\xi})(\Delta\eta + \tilde{\eta})\ \mu_{m,n+1} +$$

$$+ (\Delta\xi + \tilde{\xi})(\Delta\eta + \tilde{\eta})\ \mu_{m+1,n+1}\Big)$$

or

$$\mu(\xi,\eta) = \frac{1}{4\Delta\xi\Delta\eta}\Big((\xi_{m+1}-\xi)(\eta_{n+1}-\eta)\ \mu_{m,n} - (\xi_m-\xi)(\eta_{n+1}-\eta)\ \mu_{m+1,n} -$$

$$- (\xi_{m+1}-\xi)(\eta_n-\eta)\ \mu_{m,n+1} + (\xi_m-\xi)(\eta_n-\eta)\ \mu_{m+1,n+1}\Big)$$

for

$$\xi_m < \xi < \xi_{m+1}\ , \qquad \eta_n < \eta < \eta_{n+1}\ . \tag{III-1}$$

Now by satisfying equation (II-18) at a number of points on S_o equal to the unknown $\mu's$, the integral equation reduces to the following set of linear algebraic equations in $\mu's$:

$$\sum_{n=1}^{N} \lim_{z\to 0} \frac{\partial}{\partial z} \int_{\eta_n}^{\eta_{n+1}} \mu(0,\eta)\ d\eta \int_{-\infty}^{0} G_\zeta(x_i,y_j,z;\xi,\eta,0)\ d\xi +$$

$$+ \sum_{n=1}^{N} \sum_{m=1}^{M} \lim_{z\to 0} \frac{\partial}{\partial z} \int_{\eta_n}^{\eta_{n+1}} d\eta \int_{\xi_m}^{\xi_{m+1}} G_\zeta(x_i,y_j,z;\xi,\eta,0)\ \mu(\xi,\eta)\ d\xi d\eta$$

$$= -\delta + f_{cx}(x_i,y_j)\ . \tag{III-2}$$

To facilitate evaluation of the above integrals, we will define the following variables:

$$t = x - \xi \ ,$$

$$q = y - \eta \ ,$$

$$s = y + \eta - 2\tau \ ,$$

$$\chi(\rho,\theta) = \frac{\rho \cos^2\theta}{\gamma_o} + (y + \eta - 2\tau) \ . \qquad \text{(III-3)}$$

Then G may be written in the form

$$G_\zeta(t,q,s,z) = z/(t^2 + q^2 + z^2)^{3/2} - z/(t^2 + s^2 + z^2)^{3/2} - Re\,\frac{i}{\pi}\cdot$$
$$X \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \, d\theta \int_0^\infty e^{-\rho} \, [1/(\chi + i\omega)^2 - 1/(\chi - i\omega)^2] \; d\rho \, -$$
$$- 4\gamma_o^2 \int_1^\infty H(\bar{\omega}) \, e^{\gamma_o\lambda^2 s} \sin(\gamma_o\lambda \cdot t) \ .$$
$$X \sin(\gamma_o\lambda\sqrt{\lambda^2-1} \cdot z) \, \lambda^2 \, d\lambda$$
$$-2\gamma_o \int_1^\infty \delta(\bar{\omega}) \, e^{\gamma_o\lambda^2 s} \sin(2\gamma_o\lambda t) \, d\lambda, \qquad \text{(III-4)}$$

where

$$\omega = t \cos\theta + z \sin\theta \ ,$$

$$\bar{\omega} = - \frac{1}{\lambda}(t + \sqrt{\lambda^2-1} \cdot z) \ .$$

Also equation (III-1) may be represented as

$$\mu\left(t, \left(\begin{smallmatrix} s \\ q \end{smallmatrix}\right)\right) = \frac{1}{A} \cdot [\,\mu\,] \cdot \{Q\,{}^1_2\} \ , \qquad \text{(III-5)}$$

where in this system of notations $\left(\begin{smallmatrix} s \\ q \end{smallmatrix}\right)$ means the same equation is applicable to both the variables, s corresponds to the upper subscript and q to the lower one; A is the area of the element which is equal to $4\Delta\xi\Delta\eta$, $[\mu]$ is the row matrix

$$[\mu] = [\,\mu_{m,n}, \ \mu_{m+1,n}, \ \mu_{m,n+1}, \ \mu_{m+1,n+1}\,] \qquad \text{(III-6)}$$

and $\{Q\,\tfrac{1}{2}\}$ is the column matrix

$$\{Q\,\tfrac{1}{2}\} = \begin{Bmatrix} \mp(t_{m+1} - t)\cdot\left(\binom{s}{q}_{n+1} - \binom{s}{q}\right) \\ \pm(t_m - t)\cdot\left(\binom{s}{q}_{n+1} - \binom{s}{q}\right) \\ \pm(t_{m+1} - t)\cdot\left(\binom{s}{q}_{n} - \binom{s}{q}\right) \\ \mp(t_m - t)\cdot\left(\binom{s}{q}_{n} - \binom{s}{q}\right) \end{Bmatrix}$$

$$= \begin{bmatrix} \mp t_{m+1}\cdot\binom{s}{q}_{n+1} & \pm t_{m+1} & \pm\binom{s}{q}_{n+1} & \mp 1 \\ \pm t_m\cdot\binom{s}{q}_{n+1} & \mp t_m & \mp\binom{s}{q}_{n+1} & \pm 1 \\ \pm t_{m+1}\cdot\binom{s}{q}_{n} & \mp t_{m+1} & \mp\binom{s}{q}_{n} & \pm 1 \\ \mp t_m\cdot\binom{s}{q}_{n} & \pm t_m & \pm\binom{s}{q}_{n} & \mp 1 \end{bmatrix} \begin{Bmatrix} 1 \\ \binom{s}{q} \\ t \\ t\cdot\binom{s}{q} \end{Bmatrix}$$

It follows that (III-5) may also be expressed in the form

$$\mu\left(t,\binom{s}{q}\right) = \left[\binom{\alpha}{\beta}\right]\cdot\begin{Bmatrix} 1 \\ \binom{s}{q} \\ t \\ t\cdot\binom{s}{q} \end{Bmatrix} \qquad \text{(III-7)}$$

where $\left[\binom{\alpha}{\beta}\right]$ is the row matrix

$$\left[\binom{\alpha}{\beta}\right] = \left[\binom{\alpha}{\beta}_1, \binom{\alpha}{\beta}_2, \binom{\alpha}{\beta}_3, \binom{\alpha}{\beta}_4\right]$$

This may also be written as

$$\left[\binom{\alpha}{\beta}\right] = \frac{1}{A}\cdot[\mu]\cdot\begin{bmatrix} \mp t_{m+1}\cdot\binom{s}{q}_{n+1} & \pm t_{m+1} & \pm\binom{s}{q}_{n+1} & \mp 1 \\ \pm t_m\cdot\binom{s}{q}_{n+1} & \mp t_m & \mp\binom{s}{q}_{n+1} & \pm 1 \\ \pm t_{m+1}\cdot\binom{s}{q}_{n} & \mp t_{m+1} & \mp\binom{s}{q}_{n} & \pm 1 \\ \mp t_m\cdot\binom{s}{q}_{n} & \pm t_m & \pm\binom{s}{q}_{n} & \mp 1 \end{bmatrix}$$

(III-8)

Consider now the contribution of the wake to the integral in equation (III-2), which we will denote

by $J_w^{1,n}(x_i,y_j)$. Here

$$\mu\left(0,\ \binom{s}{q}\right) = \binom{\alpha}{\beta}_1 + \binom{\alpha}{\beta}_2 \binom{s}{q} \qquad \text{(III-9)}$$

where

$$\binom{\alpha}{\beta}_{\frac{1}{2}} = \frac{1}{2\Delta\eta} \cdot [\mu_{1,n},\ \mu_{1,n+1}] \cdot \begin{pmatrix} \pm\binom{s}{q}_{n+1} & \mp 1 \\ \mp\binom{s}{q}_n & \pm 1 \end{pmatrix} \qquad \text{(III-10)}$$

We may then write $J_w^{1,n}(x_i,y_j)$ as follows:

$$J_w^{1,n}(x_i,y_j) = \lim_{z\to 0} \frac{\partial}{\partial z} \left\{ \int_{q_n}^{q_{n+1}} (\beta_1 + \beta_2 q)\, dq \right.$$

$$X \int_\infty^{x_i} \left(z\, dt/(t^2 + q^2 + z^2)^{3/2}\right) + \int_{s_n}^{s_{n+1}} (\alpha_1 + \alpha_2 s)\, ds$$

$$X \int_\infty^{x_i} dt \{ z/(t^2 + s^2 + z^2)^{3/2} + Re\, \frac{i}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\, d\theta$$

$$X \int_0^\infty e^{-\rho} [1/(\chi + i\omega)^2 - 1/(\chi - i\omega)^2]\, d\rho + 4\gamma_o^2$$

$$X \int_1^\infty e^{\gamma_o\lambda^2 s} \left(H(\bar{\omega})\sin(\gamma_o\lambda t)\sin(\gamma_o\lambda\sqrt{\lambda^2 - 1} \cdot z) \cdot \lambda^2 + \right.$$
$$\left. + (1/2\gamma_o)\delta(\bar{\omega})\sin(2\gamma_o\lambda t)\right) d\lambda \ . \qquad \text{(III-11)}$$

This function is evaluated in Appendix B, and in reference to this may be written as

$$J_w^{1,n}(x_i,y_j) = [\beta_1,\ \beta_2] \cdot \begin{Bmatrix} b_{1,1} \\ b_{2,1} \end{Bmatrix} + [\alpha_1,\ \alpha_2] \cdot \begin{Bmatrix} b_{1,2} & b_{1,3} \\ b_{2,2} & b_{2,3} \end{Bmatrix}$$

This may now be expressed in the form

$$J_w^{1,n}(x_i,y_j) = [\mu_{1,n},\ \mu_{1,n+1}] \cdot \begin{Bmatrix} B_1^{m,n}(x_i,y_j) \\ B_2^{m,n}(x_i,y_j) \end{Bmatrix} \qquad \text{(III-12)}$$

where, from (III-10)

$$\begin{Bmatrix} B_1^{1,n}(x_i,y_j) \\ \\ B_2^{1,n}(x_i,y_j) \end{Bmatrix} = \frac{1}{2\Delta\eta}\cdot\begin{bmatrix} -q_{n+1} & 1 \\ \\ q_n & -1 \end{bmatrix}\cdot\begin{Bmatrix} b_{1,1} \\ \\ b_{2,1} \end{Bmatrix} + \frac{1}{2\Delta\eta}\cdot\begin{bmatrix} s_{n+1} & -1 \\ \\ -s_n & 1 \end{bmatrix}\begin{Bmatrix} b_{1,2}+b_{1,3} \\ \\ b_{2,2}+b_{2,3} \end{Bmatrix} \tag{III-13}$$

Consider next the contribution of the surface to the integral in equation (III-2), which we will denote by $J_s^{m,n}(x_i,y_j)$. From (III-4) and (III-7) this may be written as follows:

$$\begin{aligned} J_s^{m,n}(x_i,y_j) = \lim_{z\to 0}\frac{\partial}{\partial z}\Bigg\{ & z\int_{q_n}^{q_{n+1}} dq \int_{t_m}^{t_{m+1}} \frac{(\beta_1+\beta_2 q)+(\beta_3+\beta_4 q)\cdot t}{(t^2+q^2+z^2)^{3/2}}\,dt \\ & + \int_{s_n}^{s_{n+1}} ds \int_{t_m}^{t_{m+1}} [(\alpha_1+\alpha_2 s)+(\alpha_3+\alpha_4 s)t]\cdot\Bigg[\frac{z}{(t^2+s^2+z^2)^{3/2}} \\ & + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\, d\theta\, \mathrm{Re}\, i\int_0^\infty e^{-\rho}\left[\frac{1}{(\chi+i\omega)^2}-\frac{1}{(\chi-i\omega)^2}\right]d\rho\, + \\ & + 4\gamma_o^2\int_1^\infty e^{\gamma_o\lambda^2 s}\Big(H(\bar{\omega})\sin(\gamma_o\lambda t)\sin(\gamma\,\lambda\sqrt{\lambda^2-1}\cdot z)\cdot\lambda^2\, + \\ & + (1/2\gamma_o)\delta(\bar{\omega})\sin(2\gamma_o\lambda t)\Big)\,d\lambda\Bigg]\,dt.\Bigg\} \end{aligned} \tag{III-14}$$

It turns out (see Appendix C) that this expression may be written as

$$J_s^{m,n}(x_i,y_j) = [\beta_\kappa]^{\cdot}\{a_{\kappa,1}\} + [\alpha_\kappa]^{\cdot}\{a_{\kappa,2}+a_{\kappa,3}+a_{\kappa,4}+a_{\kappa,5}\}$$

$$\text{and } \kappa = 1,2,3,4 \quad .$$

Substituting for $\binom{\beta}{\alpha}_\kappa$ from (III-8), we may express $J_s^{m,n}(x_i,y_j)$ in the form

$$J_s^{m,n}(x_i,y_j) = [\mu]\cdot\begin{Bmatrix} A_1^{m,n}(x_i,y_j) \\ A_2^{m,n}(x_i,y_j) \\ A_3^{m,n}(x_i,y_j) \\ A_4^{m,n}(x_i,y_j) \end{Bmatrix} \tag{III-15}$$

where

$$\begin{Bmatrix} A_1^{m,n}\ (x_i,y_j) \\ A_2^{m,n}\ (x_i,y_j) \\ A_3^{m,n}\ (x_i,y_j) \\ A_4^{m,n}\ (x_i,y_j) \end{Bmatrix} = \frac{1}{A} \cdot \begin{bmatrix} t_{m+1}\ q_{n+1} & -t_{m+1} & -q_{n+1} & 1 \\ -t_m\ \cdot\ q_{n+1} & t_m & q_{n+1} & -1 \\ -t_{m+1}\cdot q_n & t_{m+1} & q_n & -1 \\ t_m\ q_n & -t_m & -q_n & 1 \end{bmatrix} \cdot \begin{Bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ a_{4,1} \end{Bmatrix}$$

$$+\frac{1}{A} \cdot \begin{bmatrix} -t_{m+1}\ s_{n+1} & t_{m+1} & s_{n+1} & -1 \\ t_m\ s_{n+1} & -t_m & -s_{n+1} & 1 \\ t_{m+1}\ s_n & -t_{m+1} & -s_n & 1 \\ -t_m\ .\ s_n & t_m & s_n & -1 \end{bmatrix} \cdot \begin{Bmatrix} a_{1,2} + a_{1,3} + a_{1,4} + a_{1,5} \\ a_{2,2} + a_{2,3} + a_{2,4} + a_{2,5} \\ a_{3,2} + a_{3,3} + a_{3,4} + a_{3,5} \\ a_{4,2} + a_{4,3} + a_{4,4} + a_{4,5} \end{Bmatrix} \qquad \text{(III-16)}$$

We may now write equation (III-2) in the form

$$\sum_{n=1}^{N} [\,\mu_{1,n},\mu_{1,n+1}\,] . \begin{Bmatrix} B_1^{1,n}\ (x_i,y_j) \\ B_2^{1,n}\ (x_i,y_j) \end{Bmatrix} + \sum_{n=1}^{N}\sum_{m=1}^{M} [\mu] \cdot \left\{A_\kappa^{m,n}\ (x_i,y_j)\right\}$$

$$= \delta - f_{\alpha x}\ (x_i,y_j)\ , \qquad \kappa = 1,2,3,4\ . \qquad \text{(III-17)}$$

The left-hand side of this equation can be written as

$$\sum_{n=1}^{N} \Big\{ B_1^{1,n}\ (x_i,y_j)\ \mu_{1,n} + B_2^{1,n}\ (x_i,y_j)\ \mu_{1,n+1} +$$

$$+ \sum_{m=1}^{M} \left(A_1^{m,n}\ (x_i,y_j)\ \mu_{m,n} + A_3^{m,n}\ (x_i,y_j)\ \mu_{m,n+1}\right) +$$

$$+ \sum_{\kappa=2}^{M+1} \left(A_2^{\kappa-1,n}\ (x_i,y_j)\ \mu_{\kappa,n} + A_4^{\kappa-1,n}\ (x_i,y_j)\ \mu_{\kappa,n+1}\right)\Big\}$$

$$+ \sum_{n=1}^{N} \Big\{ [\,B_1^{1,n}\ (x_i,y_j) + A_1^{1,n}\ (x_i,y_j)\,]\ \mu_{1,n} + A_2^{m,n}\ (x_i,y_j)\ \mu_{M+1,n} +$$

$$+ \sum_{m=2}^{M} A_1^{m,n}(x_i,y_j) + A_2^{m-1,n}(x_i,y_j) \quad \mu_{m,n}\Big\} +$$

$$+ \sum_{\kappa=2}^{N+1} \Big\{ B_2^{1,\kappa-1}(x_i,y_j) + A_3^{1,\kappa-1}(x_i,y_j) \quad \mu_{1,\kappa} + A_4^{M,\kappa-1}(x_i,y_j) \quad {}_{M,\kappa} +$$

$$+ \sum_{m=2}^{M} A_3^{m,\kappa-1}(x_i,y_j) + A_4^{m-1,\kappa-1}(x_i,y_j) \quad \mu_{m,\kappa}\Big\}$$

$$= \sum_{n=1}^{N} \Big\{ B_1^{1,n}(x_i,y_j) + A_1^{1,n}(x_i,y_j) \quad \mu_{1,n} + A_2^{M,n}(x_i,y_j) \quad \mu_{M+1,n}\Big\} +$$

$$+ \sum_{n=2}^{N+1} \Big\{ B_2^{1,n-1}(x_i,y_j) + A_3^{1,n-1}(x_i,y_j) \quad \mu_{1,n} + A_4^{M,n-1}(x_i,y_j)\mu_{M+1,n}\Big\} +$$

$$+ \sum_{m=2}^{M} \Big\{ A_1^{m,1}(x_i,y_j) + A_2^{m-1,1}(x_i,y_j) \quad \mu_{m,1} +$$

$$+ A_3^{m,N}(x_i,y_j) + A_4^{m-1,N}(x_i,y_j) \quad \mu_{m,N+1}\Big\} +$$

$$+ \sum_{n=2}^{N} \sum_{m=2}^{M} A_1^{m,n}(x_i,y_j) + A_2^{m-1,n}(x_i,y_j) +$$

$$+ A_3^{m,n-1}(x_i,y_j) + A_4^{m-1,n-1}(x_i,y_j) \quad \mu_{m,n} \quad .$$

From equation (II-6), we find that

$$\mu_{m,1} = 0 \quad , \qquad\qquad m = 1,2,\ldots,M+1 \quad .$$

Finally, we can write the set of the linear algebraic equations for $\mu_{m,n}$ in the form

$$\sum_{n=2}^{N+1} \sum_{m=1}^{M+1} C_{m,n}(x_i,y_j)\ \mu_{m,n} = f_{\alpha x}(x_i,y_j) - \delta \qquad \text{(III-18)}$$

where

$$C_{m,n}(x_i,y_j) = A_1^{m,n}(x_i,y_j) + A_2^{m-1,n}(x_i,y_j) +$$

$$+ A_3^{m,n-1}(x_i,y_j) + A_4^{m-1,n-1}(x_i,y_j) \quad ,$$

$$m = 2,3,\ldots,M \quad , \qquad n = 2,3,\ldots,N \quad ;$$

$$C_{m,N+1}(x_i,y_j) = A_3^{m,N}(x_i,y_j) + A_4^{m-1,N}(x_i,y_j) \quad ,$$

$$m = 2,3,\ldots,M \quad ;$$

$$C_{M+1,n}(x_i,y_j) = A_2^{M,n}(x_i,y_j) + A_4^{M,n-1}(x_i,y_j) \quad ,$$

$$n = 2,3,\ldots,N \quad ;$$

$$C_{1,n}(x_i,y_j) = B_1^{1,n}(x_i,y_j) + B_2^{1,n-1}(x_i,y_j) +$$

$$+ A_1^{1,n}(x_i,y_j) + A_3^{1,n-1}(x_i,y_j) \quad ,$$

$$n = 2,3,\ldots,N \quad ;$$

$$C_{M+1,N+1}(x_i,y_j) = A_4^{M,N}(x_i,y_j) \quad ;$$

$$C_{1,N+1}(x_i,y_j) = B_2^{1,N}(x_i,y_j) + A_3^{1,N}(x_i,y_j) \quad .$$

(III-19)

Equation (III-18) can now be solved numerically for a given body moving with a constant velocity u_o at an angle of attack δ .

IV. Numerical Results and Conclusions

A computer program was developed, based on the numerical analysis given in Appendix D, to compute the moment distribution μ as well as the forces and moments acting on a yawed and/or cambered body.

The numerical scheme was first applied to the case of zero speed, where the free surface is regarded as a rigid wall and the flow is the same as that around a fully submerged double body consisting of the ship and its image over the free surface. The curves in Figures 5 and 6 show the results of such computations and are compared to similar results given by Thwaites (1960, p. 343) for rectangular wings.

The effect of the free surface was then considered for a small Froude number ($F_n = u_o/\sqrt{gL} = 0.1$) and the computations were made for two different values of the length/draft ratio, L/T=7 and L/T=20. The results showed a very slight deviation from the values obtained for the case of a rigid surface.

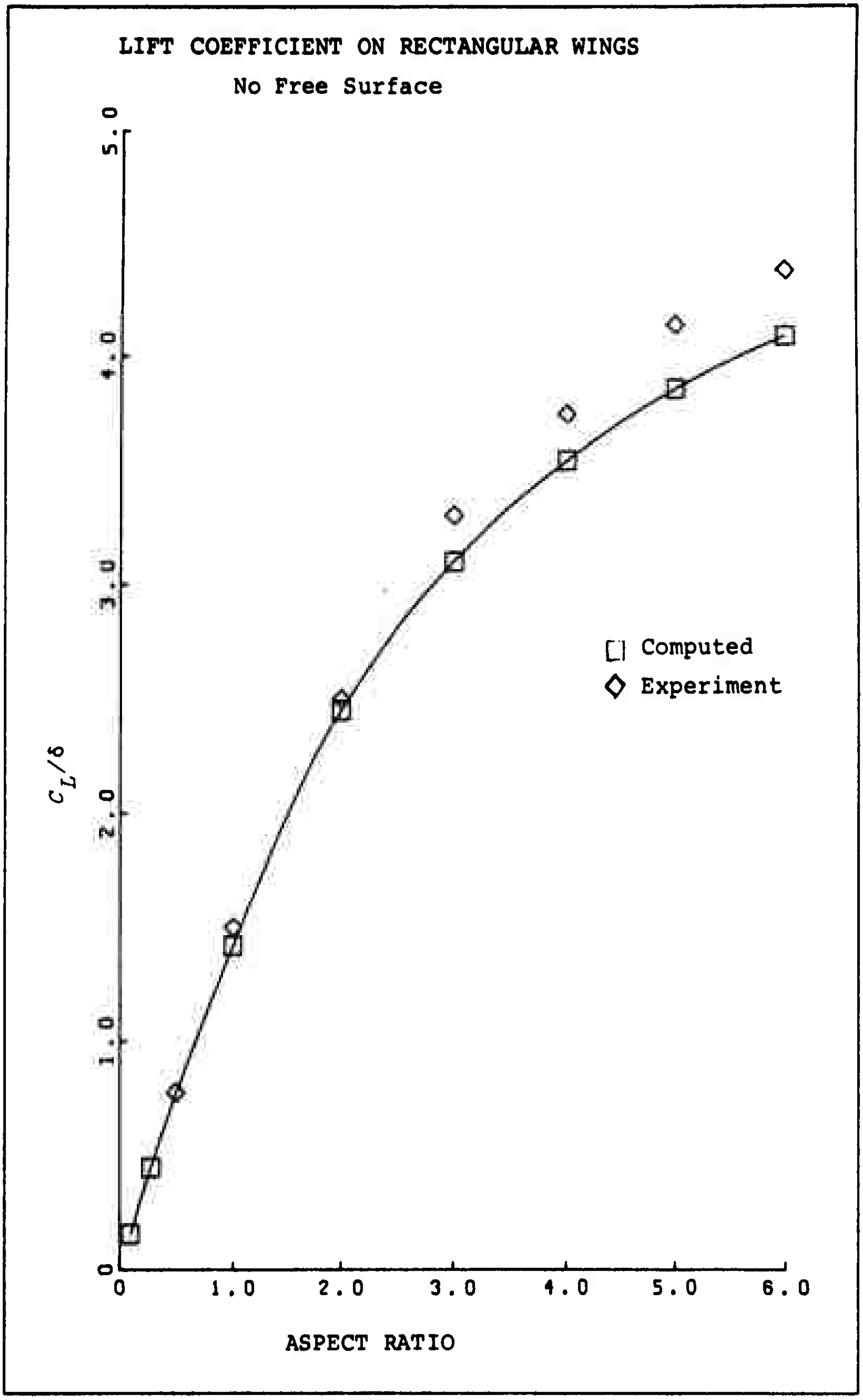
Computations were carried out for a yawed body having a length/draft ratio equal to 20 at different Froude numbers and two values were used for the number of nodal points in the longitudinal direction, M=5 and M=10. Also one run was made for the same body with M=20 and F_n=0.34. The results of these computations are shown in Figures 7-11.

There is a large discrepancy between the results obtained when M is set equal to 10 and those obtained for $M=5$ and $M=20$ in the range of Froude numbers between $F_n=0.225$ and $F_n=0.45$. Norrbin (1960, p. 379) pointed out that "for a surface ship running at Froude numbers exceeding $F_n = u_o/\sqrt{gL} = 0.3$, wave formation is generally found to cause a change in trim and stability characteristics." Hu (1961) found out that the lateral stability derivatives have increased about 50% above their values at zero speed for a Froude number $F_n=0.35$. Due to the lack of experimental data at higher Froude numbers, it is difficult to predict the behavior of the force and moment coefficients in this region. On the other hand, there is the possibility that the numerical analysis is divergent for the particular value $M=10$ in the range of Froude numbers where this behavior is encountered.

Conclusions

Although one of the purposes of this study has been to investigate theoretically the side forces and moments acting on a yawed body in a free surface, an equally important one has been the testing of the applicability of a numerical method for solving the complicated integral equation that arises. The agreement

of the results obtained by this method with those obtained by other methods when the Froude number is small seems to indicate that the method is fundamentally sound. For higher Froude numbers the inconsistency of the values for N=10 with those for N=5 and N=20 indicates that further investigation of numerical stability in this region is necessary. Further experimental work supplementing Norrbin's measurements is also desirable.



LIFT COEFFICIENT ON RECTANGULAR WINGS
No Free Surface
C_L/δ
0
1.0
2.0
3.0
4.0
5.0
0
1.0
2.0
3.0
4.0
5.0
6.0
ASPECT RATIO
Computed
Experiment

Figure 5

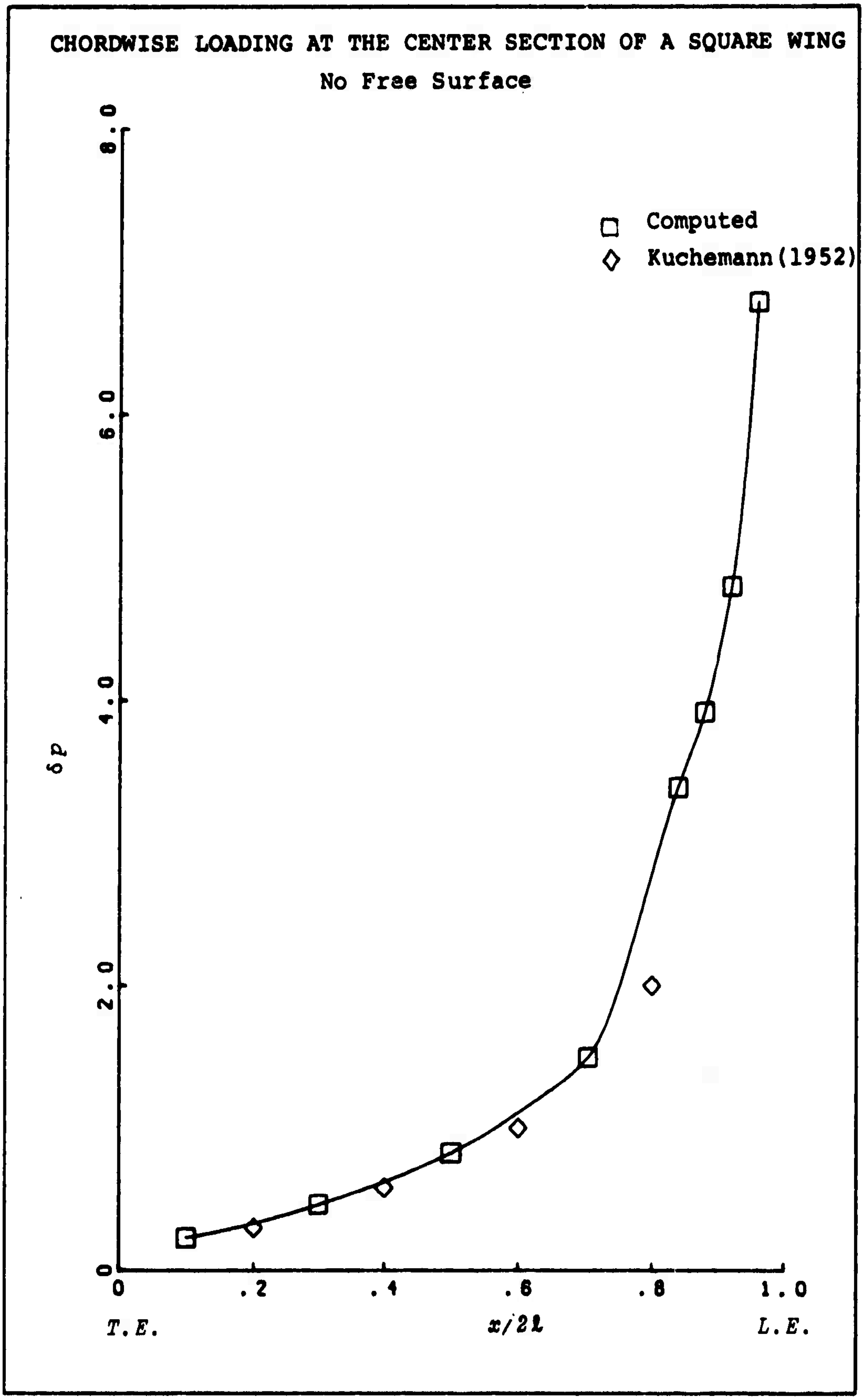

CHORDWISE LOADING AT THE CENTER SECTION OF A SQUARE WING
No Free Surface
Computed
Kuchemann(1952)
δp
0
2.0
4.0
6.0
8.0
0
.2
.4
.6
.8
1.0
T.E.
x/2ℓ
L.E.

Figure 6

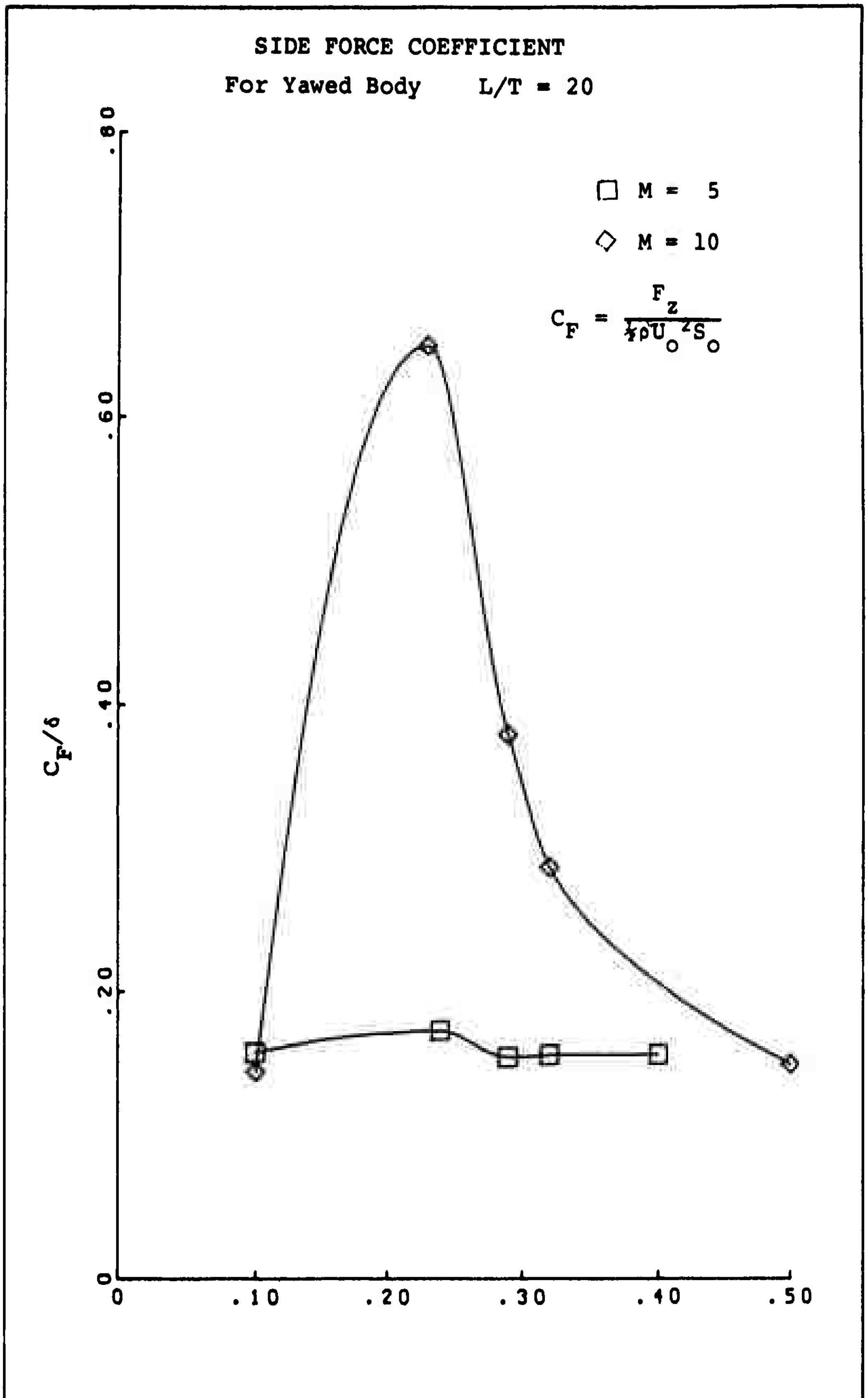

SIDE FORCE COEFFICIENT
For Yawed Body L/T = 20
M = 5
M = 10
$C_F = \frac{F_z}{\frac{1}{2}\rho U_o^2 S_o}$
C_F/δ
0
.20
.40
.60
.80
0
.10
.20
.30
.40
.50

Figure 7

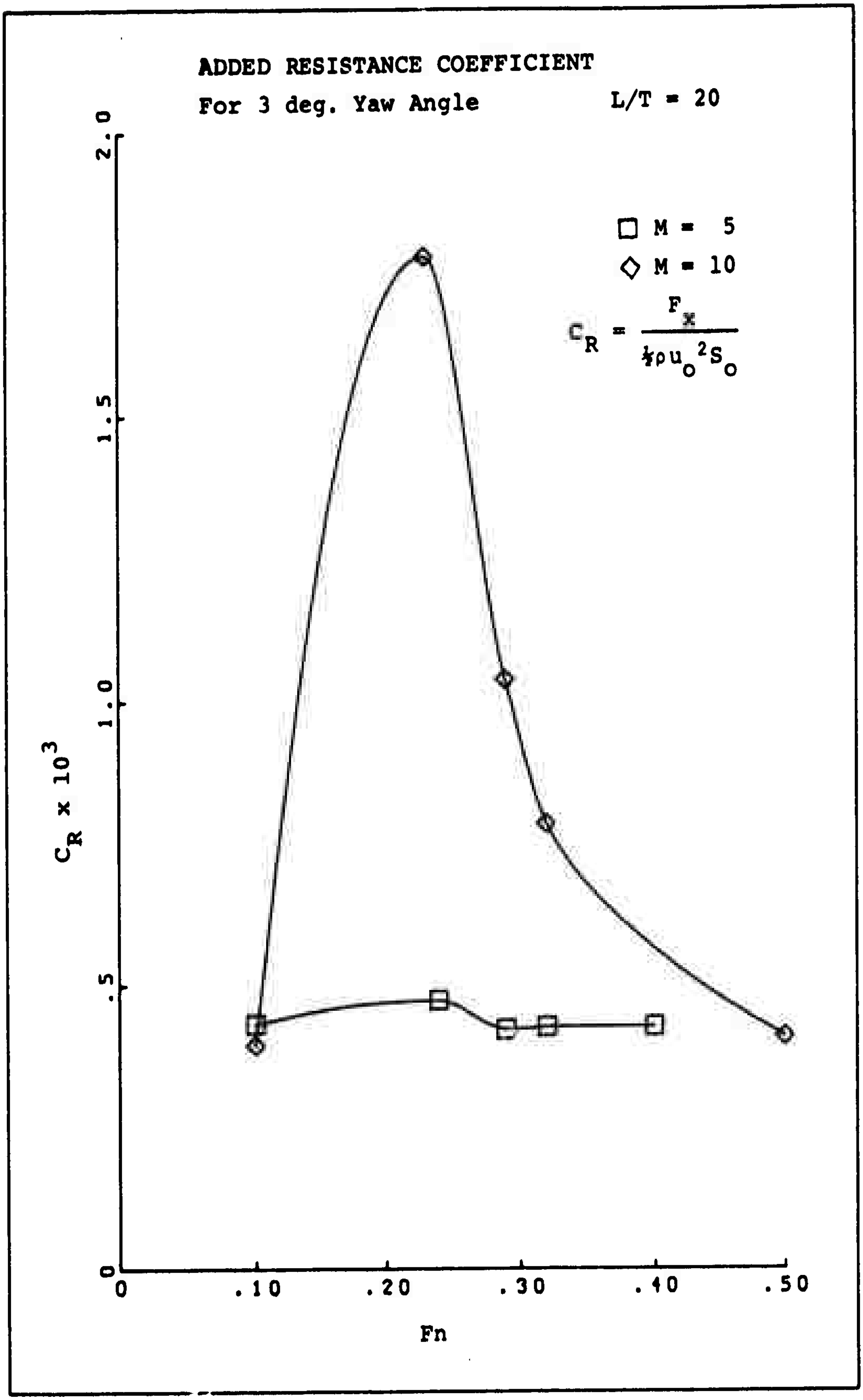
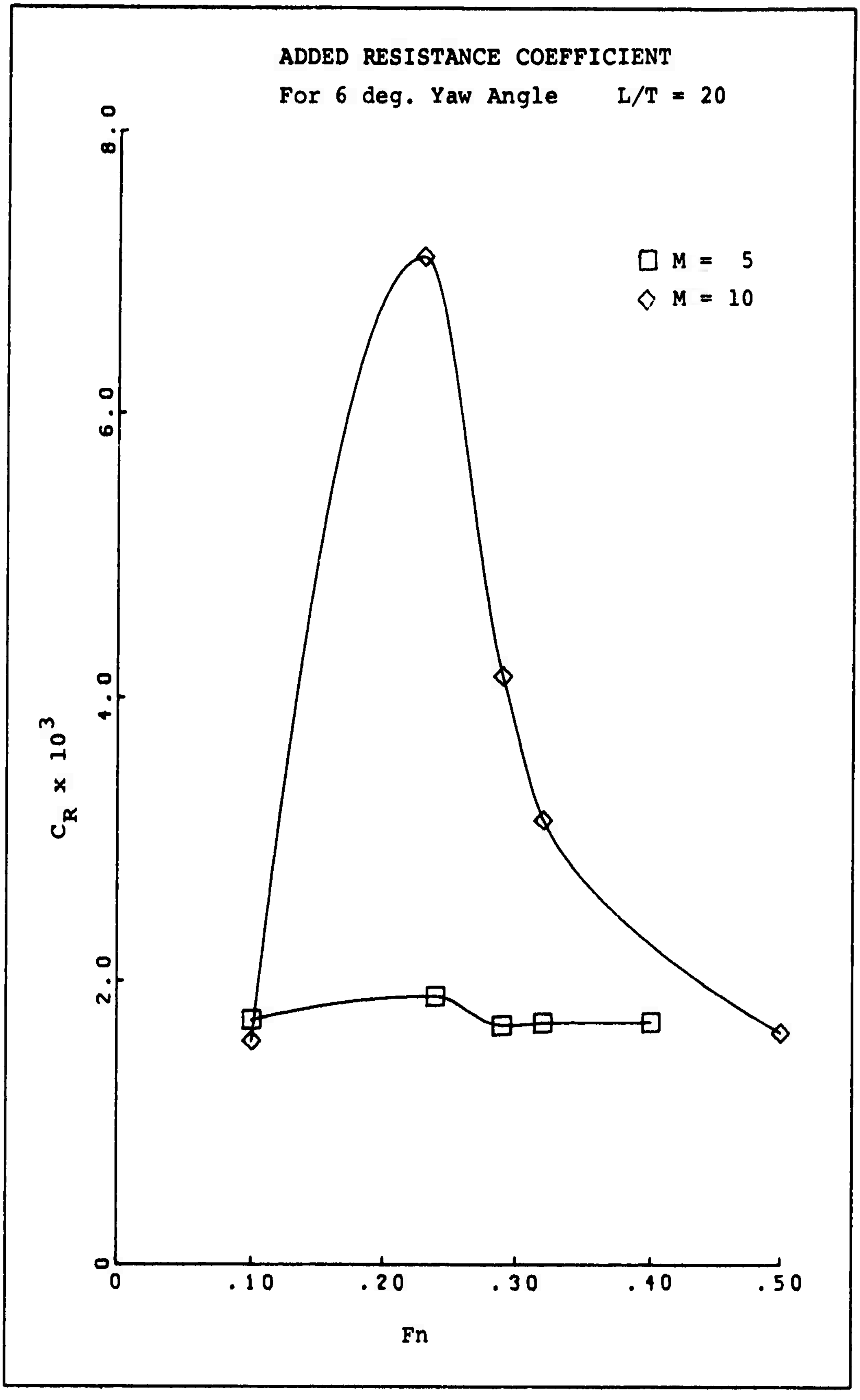

ADDED RESISTANCE COEFFICIENT
For 3 deg. Yaw Angle
L/T = 20
M = 5
M = 10
$C_R = \frac{F_x}{\frac{1}{2}\rho u_o^2 S_o}$
$C_R \times 10^3$
2.0
1.5
1.0
.5
0
0
.10
.20
.30
.40
.50
Fn

Figure 8



ADDED RESISTANCE COEFFICIENT
For 6 deg. Yaw Angle L/T = 20
M = 5
M = 10
$C_R \times 10^3$
0
2.0
4.0
6.0
8.0
0
.10
.20
.30
.40
.50
Fn

Figure 9

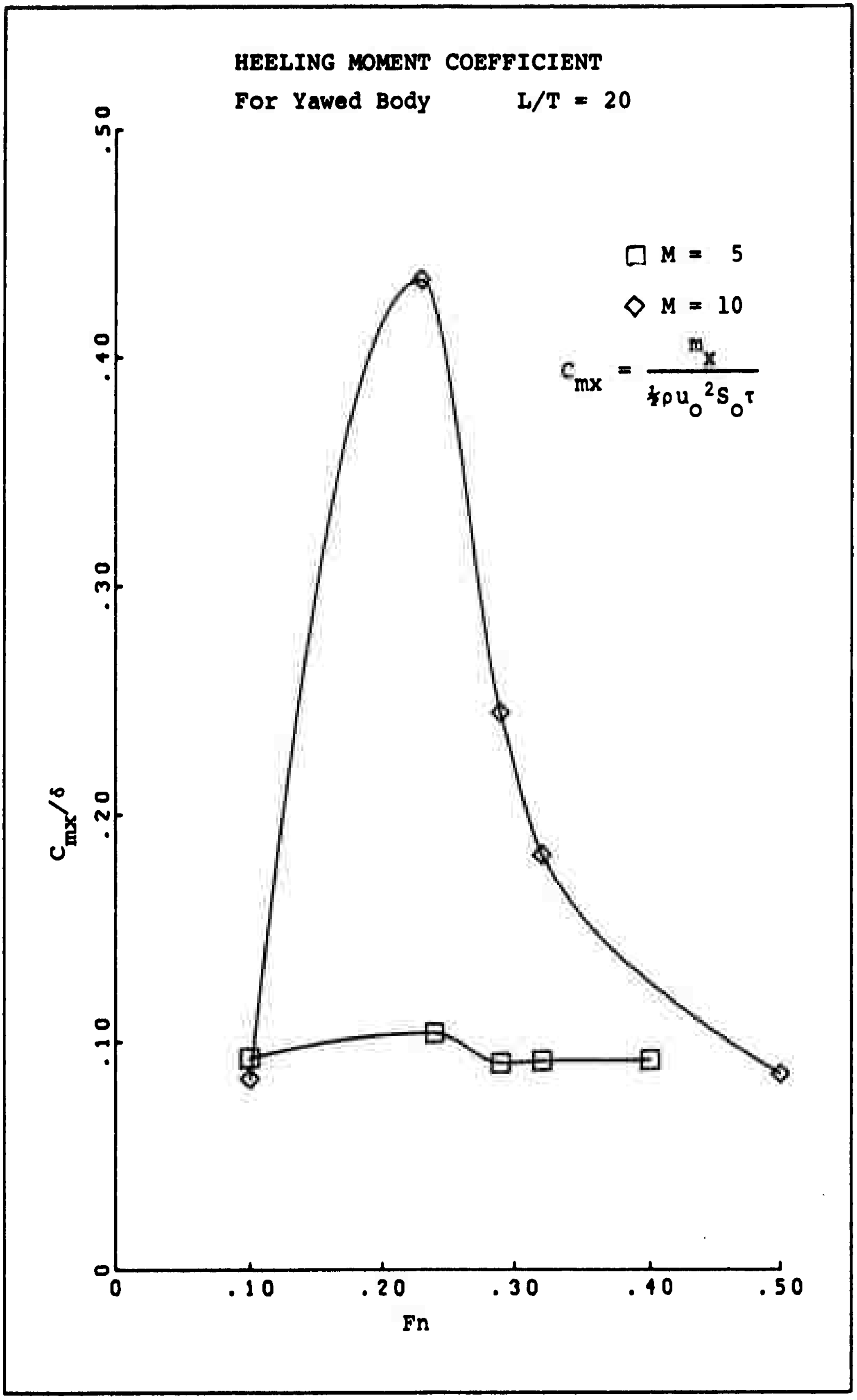

HEELING MOMENT COEFFICIENT
For Yawed Body L/T = 20
□ M = 5
◇ M = 10
$C_{mx} = \frac{m_x}{\frac{1}{2}\rho u_o^2 S_o \tau}$
C_{mx}/δ
0
.10
.20
.30
.40
.50
Fn

Figure 10

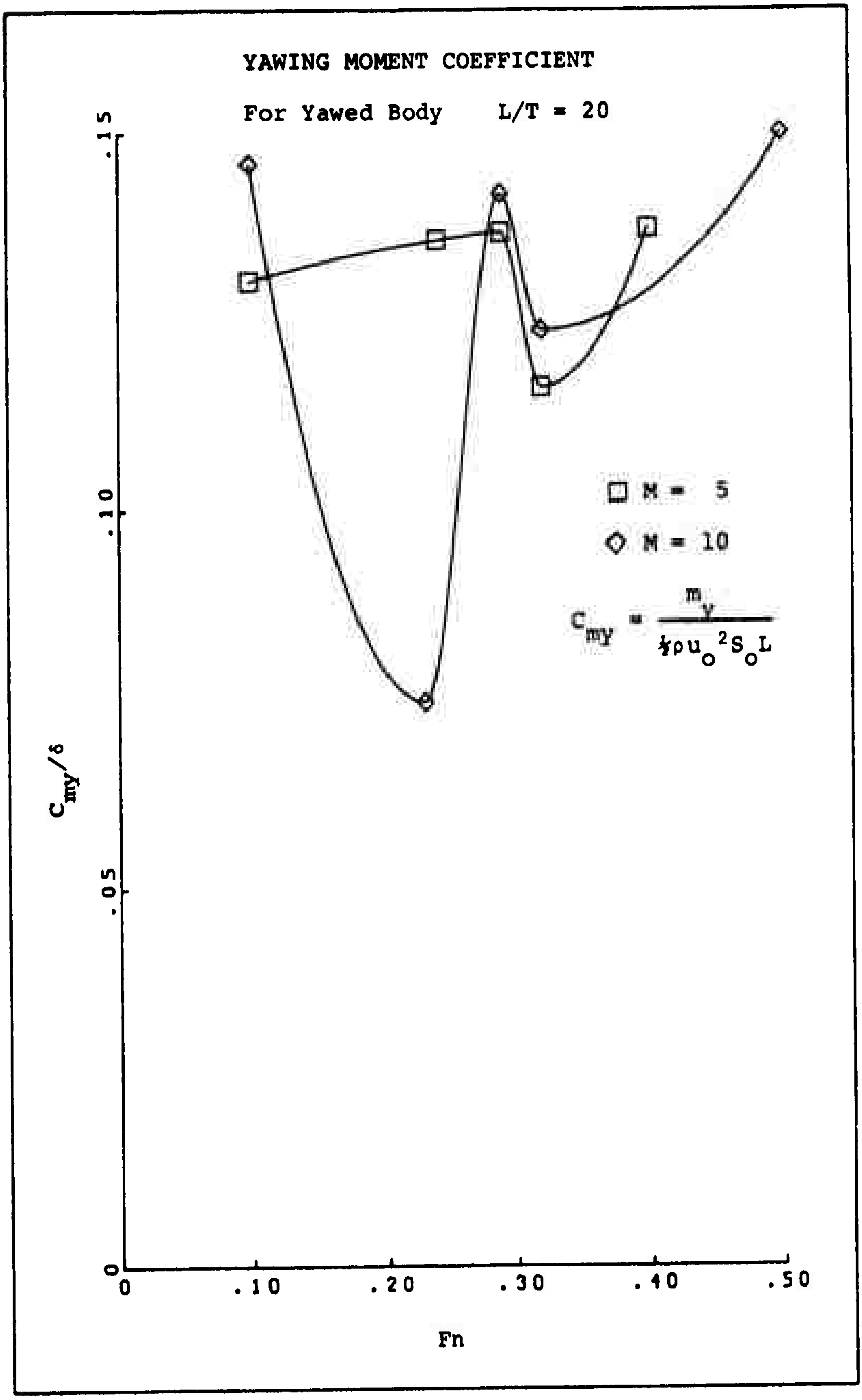

YAWING MOMENT COEFFICIENT
For Yawed Body L/T = 20
□ M = 5
◇ M = 10
$C_{my} = \frac{m_y}{\frac{1}{2}\rho u_o^2 S_o L}$
C_{my}/δ
.15
.10
.05
0
0
.10
.20
.30
.40
.50
Fn

Figure 11

ACKNOWLEDGMENTS

The author wishes to express herein his deep gratitude to all those, at both ends of the hemisphere, who have helped him in his work. Above all, his great indebtedness goes to Professor John V. Wehausen who throughout has counseled him with patience and understanding, as well as encouragement. He also would like to thank Professor W. C. Webster for his advice in the course of this work, and Professor S. A. Berger for reviewing the manuscript.

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BIBLIOGRAPHY

Abramowitz, M.; Segun, Irene A.

Handbook of mathematical functions. Dover Publ., Inc., New York, 1968, xiv+1046 pp.

Ashley, H.; Landahl, M.

Aerodynamics of wings and bodies. Addison-Wesley Publ., Inc., Massachusetts, 1965, vii+279 pp.

Chen, Hsao-Hsin.

The experimental determination of the rudder forces and rudder torque of a mariner class ship model. Univ. of Calif., Berkeley, Coll. of Engrg., Rep. No. NA-67-5, July 1967, 30 pp.

Davidson, K. S. M.; Schiff, L. I.

Turning and course keeping qualities. Trans. Soc. Naval Arch. Mar. Eng. 54(1946), pp. 152-200.

Eggers, K. W. H.; Sharma, S. D.; Ward, L. W.

An assessment of some experimental methods for determining the wave-making characteristics of a ship form. Trans. Soc. Naval Arch. Mar. Eng. 75(1967), pp. 112-157.

Hu, Pung Nien.

Forward speed effect on lateral stability derivatives of a ship. Davidson Lab., Rep. No. 829 (1961), iii+35 pp.

Kellogg, O. D.

Foundations of potential theory. Dover Publ., Inc., New York, 1929, ix+384 pp.

Martin, M.

Analysis of lateral force and moment caused by yaw during ship turning. Davidson Lab. Rep. No. 792, (1961).

Newman, J. N.

A linearized theory for the motion of a thin ship in regular waves. JSR, 5 (1961), pp. 34-55.

Some hydrodynamic aspects of ship maneuverability. Sixth Symposium Naval Hydrodynamics (1966), pp. 203-237.

Norrbin, Nils H.

A study of course keeping and maneuvering performance. First Symposium on Ship Maneuvering, DTMB, Rep. No. 1461 (1960), pp. 359-423.

Robinson, A.; Lausmann, J. A.

Wing theory. Cambridge Aeronautical Series, vol. II, University Press, Cambridge, 1956, ix+569 pp.

Thwaites, Bryan.

Incompressible aerodynamics. Fluid Motion Memoirs, Clarendon Press, Oxford, 1960, xx+636 pp.

Van Dyke, M.

Perturbation methods in fluid mechanics. Academic Press, New York, 1964, x+229 pp.

Wehausen, John V.

An approach to thin-ship theory. Internat. Sem. Theorit. Wave Resistance, Ann Arbor, 1963, pp. 819-852; disc. 853-855.

Wehausen, John V.; Laitone, E. V.

Surface waves. Encyclopedia of Physics, vol. IX, pp. 446-778. Springer-Verlag, Berlin, 1960.

Wieghardt, Karl.

Chordwise load distribution of a simple rectangular wing. N.A.C.A., Tech. Mem., Washington, 1940, 28 pp.

Yeung, Ronald W.

Sinkage and trim in first-order thin-ship theory. JSR, 16 (1972).

Zienkiewicz, O. Z.

Finite element method in structural and continuum mechanics. McGraw-Hill, London, 1971, xiv+521 pp.

Appendix A

Green function $G(x,y,z;\xi,\eta,\zeta)$ is given by equation (II-9). Consider now the integral with respect to κ in this function, which we will write in the form

$$\int_{c_1} \frac{d\kappa}{\kappa - \gamma_o sec^2\theta}\, exp\,[-\kappa a e^{-i\psi}] + \int_{c_2} \frac{d\kappa}{\kappa - \gamma_o sec^2\theta}\, exp\,[-\kappa a e^{i\psi}] \tag{A-1}$$

where

$$-ae^{-i\psi} = (y + \eta) + i\omega \;, \qquad a > 0 \;.$$

Therefore $\quad -a\, cos\psi = y + \eta < 0 \;, \qquad -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$

and $\quad \omega = a\, sin\psi \;.$

Let us evaluate the above integrals over the straight line $\kappa = re^{i\phi_o}$ in the complex κ - plane. Then we obtain

$$\int_{c_1} \frac{dre^{i\phi_o}}{re^{i\phi_o} - \gamma_o sec^2\theta}\, exp\,[-are^{i(\phi_o-\psi)}] + \int_{c_2} \frac{dre^{i\phi_o}}{re^{i\phi_o} - \gamma_o sec^2\theta}\, exp\,[-are^{i(\phi_o+\psi)}]$$

The oscillating part in these integrals will vanish if we set $\phi_o = \psi$ in the first one and $\phi_o = -\psi$ in the second. Then it becomes

$$\int_{c_1'} \frac{e^{-ar}}{r - \gamma_o sec^2\theta e^{-i\psi}}\, dr + \int_{c_2'} \frac{e^{-ar}}{r - \gamma_o sec^2\theta e^{i\psi}}\, dr =$$

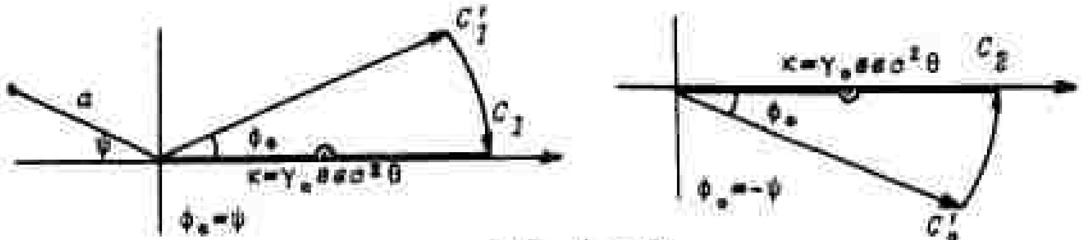
$$= \int_{c_1'} \frac{e^{-\rho}}{\rho + \rho_1}\, d\rho + \int_{c_2'} \frac{e^{-\rho}}{\rho + \rho_2}\, d\rho \tag{A-2}$$

where

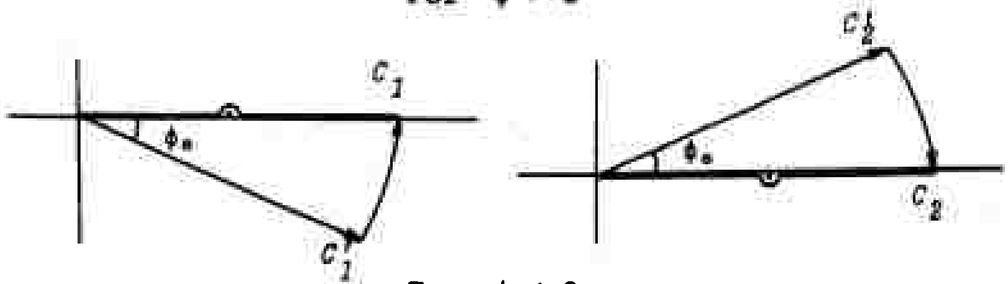
$$\rho_1 = \gamma_o sec^2\theta\, [(y + \eta) + i\omega]$$

$$\rho_2 = \gamma_o sec^2\theta\, [(y + \eta) - i\omega]$$

We will apply now Cauchy's Theorem for contour integration to evaluate the integrals in (A-1). The paths of integration will be completed as shown in the figures below to form a closed contour.



For $\psi > 0$



For $\psi < 0$

In order that the integrals along the arc converge to zero as $R \to \infty$, it is necessary and sufficient that $-\pi/2 \leq \phi_o \pm \psi \leq \pi/2$. This requirement is satisfied by the previous choice of ϕ_o.

If $\bar{\omega} < 0$, we must take into account the residue at the pole at $\kappa = \gamma_o sec^2\theta$; if $\bar{\omega} > 0$, there is no residue. By using the result given in (A-2), the integral in (A-1) can be written as

$$\int_0^\infty e^{-\rho} \left[\frac{1}{\rho \mp \rho_1} + \frac{1}{\rho \mp \rho_2}\right] d\rho + H(-\bar{\omega}) \{Residue\}$$

where $H(-\bar{\omega})$ is the Heavyside function.

Finally we will now evaluate the residue at $\kappa = \gamma_o \sec^2\theta$ as follows:

$$-Re\,\frac{\gamma_o}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\,\sec^2\theta\,\pi i\Big[-e^{\gamma_o\sec^2\theta\,[(y+\eta)+i\omega]}+ e^{\gamma_o\sec^2\theta\,[(y+\eta)-i\omega]}\Big]$$

$$= Im\,\gamma_o\Bigg\{\int_{-\frac{\pi}{2}}^{0} d\theta\,\sec^2\theta\Big[e^{\gamma_o\sec^2\theta\,[(y+\eta)+i\omega]} - e^{\gamma_o\sec^2\theta\,[(y+\eta)-i\omega]}\Big] + \int_0^{\frac{\pi}{2}} d\theta\,\sec^2\theta\Big[e^{\gamma_o\sec^2\theta\,[(y+\eta)+i\omega]} - e^{\gamma_o\sec^2\theta\,[(y+\eta)-i\omega]}\Big]\Bigg\}$$

$$= Im\,2\gamma_o\int_0^{\frac{\pi}{2}} d\theta\,\sec^2\theta\,e^{\gamma_o\sec^2\theta\,(y+\eta)}\cdot\cos[\gamma_o(z-\zeta)\sin\theta\sec^2\theta]\;X\;[e^{i\gamma_o\sec\theta\,(x-\xi)} - e^{-i\gamma_o\sec\theta\,(x-\xi)}]$$

$$= -4\gamma_o\int_0^{\frac{\pi}{2}} e^{\gamma_o\sec^2\theta\,(y+\eta)}\cdot\sin[\gamma_o\sec\theta\,(x-\xi)]\;X\cos[\gamma_o(z-\zeta)\sin\theta\sec^2\theta]\sec^2\theta\,d\theta \qquad \text{(A-3)}$$

Appendix B

The $J_w^{1,n}(x_i,y_j)$ Integral

This integral is defined by equation (III-11). We may write it in the following form:

$$J_w^{1,n}(x_i,y_j) = \lim_{z\to 0} \frac{\partial}{\partial z} \Bigg[-z \int_{q_n}^{q_{n+1}} (\beta_1 + \beta_2 q)\, dq$$

$$X \int_{x_i}^{\infty} dt/(t^2 + q^2 + z^2)^{3/2} \; -$$

$$-z \int_{s_n}^{s_{n+1}} (\alpha_1 + \alpha_2 s)\, ds \int_{x_i}^{\infty} dt/(t^2 + s^2 + z^2)^{3/2} \; -$$

$$-\frac{1}{\pi} \int_{x_i}^{\infty} dt \int_{s_n}^{s_{n+1}} (\alpha_1 + \alpha_2 s)\, ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\, d\theta$$

$$X \int_0^{\infty} e^{-\rho} \cdot Re \left[\frac{i}{(\chi + i\omega)^2} - \frac{i}{(\chi - i\omega)^2}\right] d\rho \; +$$

$$+ 2\gamma_o \int_{x_i}^{\infty} dt \int_{s_n}^{s_{n+1}} (\alpha_1 + \alpha_2 s)\, ds \int_1^{\infty} e^{\gamma_o \lambda^2 z} \Big(2\gamma_o H(\overline{\omega})$$

$$X \sin(\gamma_o \lambda t) \sin(\gamma_o \lambda \sqrt{\lambda^2 - 1} \cdot z) \cdot \lambda^2 + \delta(\overline{\omega}) \sin(2\gamma_o \lambda t)\Big)\, d\lambda \Bigg]$$

(B-1)

Now we will evaluate each of the above integrals in (B-1) separately and determine the limit of its derivative with respect to z as $z \to 0$. Consider the first integral, which we will write as

$$J_1 = -z \int_{q_n}^{q_{n+1}} (\beta_1 + \beta_2 q)\, dq \int_{x_i}^{\infty} \frac{dt}{(t^2 + q^2 + z^2)^{3/2}}$$

After integrating this once over t, we have

$$J_1 = -z \int_{q_n}^{q_{n+1}} (\beta_1 + \beta_2 q) \cdot \left[\frac{1}{q^2 + z^2} \left(1 - \frac{x_i}{\sqrt{q^2 + x_i^2 + z^2}}\right)\right] dq$$

$$= -z \left\{\beta_1 \cdot \int_{q_n}^{q_{n+1}} \left[\frac{1}{q^2 + z^2} - \frac{x_i}{(q^2 + z^2)\sqrt{q^2 + x_i^2 + z^2}}\right] dq + \right.$$

$$\left. + \beta_2 \int_{q_n}^{q_{n+1}} \left[\frac{q}{q^2 + z^2} - \frac{x_i q}{(q^2 + z^2)\sqrt{q^2 + x_i^2 + z^2}}\right] dq \right\} .$$

The integration of the second term in each of the above integrals can be easily performed if we make the substitutions $v_1 = \frac{q}{\sqrt{q^2 + x_i^2 + z^2}}$ in the first and $v_2 = \sqrt{q^2 + x_i^2 + z^2}$ in the second. Thus

$$J_1 = \beta_1 \left\{ \left[\tan^{-1} \frac{q_n}{z} - \tan^{-1} \frac{q_{n+1}}{z}\right] + \right.$$

$$\left. + \left[\tan^{-1} \frac{x_i q_{n+1}}{z\sqrt{z^2 + x_i^2 + q_{n+1}^2}} - \tan^{-1} \frac{x_i q_n}{z\sqrt{z^2 + x_i^2 + q_n^2}}\right]\right\} +$$

$$+ \beta_2 \left\{ \frac{z}{2} \left[\ln(q_n^2 + z^2) - \ln(q_{n+1}^2 + z^2)\right] - \right.$$

$$\left. - z \left[\coth^{-1} \frac{\sqrt{z^2 + x_i^2 + q_{n+1}^2}}{x_i} - \coth^{-1} \frac{\sqrt{z^2 + x_i^2 + q_n^2}}{x_i}\right]\right\} .$$

The contribution of J_1 to $J_w^{1,n}(x_i,y_j)$ is

$$\lim_{z\to 0} \frac{\partial J_1}{\partial z} = \beta_1 \left\{\left(\frac{1}{q_{n+1}} - \frac{1}{q_n}\right) - \frac{1}{x_i}\left[\sqrt{q_{n+1}^2 + x_i^2}/q_{n+1} - \sqrt{q_n^2 + x_i^2}/q_n\right]\right\} +$$

$$+ \beta_2 \left\{\ln \frac{q_n}{q_{n+1}} - \left[\coth^{-1} \left(\sqrt{q_{n+1}^2 + x_i^2}/x_i\right) - \right.\right.$$

$$\left.\left. - \coth^{-1} \left(\sqrt{q_n^2 + x_i^2}/x_i\right)\right]\right\} .$$

Define

$$Q^{i,j} = \sqrt{q^2 + x_i^2} \tag{B-2}$$

We may then write the above equation as

$$\lim_{z\to 0} \frac{\partial J_1}{\partial z} = [\beta_1, \beta_2] \cdot \begin{Bmatrix} b_{1,1} \\ b_{2,2} \end{Bmatrix} , \tag{B-3}$$

where

$$b_{1,1} = \left(\frac{1}{q_{n+1}} - \frac{1}{q_n}\right) - \frac{1}{x_i}\left(\frac{Q_{n+1}^{i,j}}{q_{n+1}} - \frac{Q_n^{i,j}}{q_n}\right),$$

$$b_{2,1} = \ln\left(\frac{|q_n|}{|q_{n+1}|}\right) - \left[\coth^{-1}\left(\frac{Q_{n+1}^{i,j}}{x_i}\right) - \coth^{-1}\left(\frac{Q_n^{i,j}}{x_i}\right)\right] . \tag{B-4}$$

The second integral in (B-1) may be written as

$$J_2 = -z \int_{s_n}^{s_{n+1}} (\alpha_1 + \alpha_2 s)\, ds \int_{x_i}^{\infty} \frac{dt}{(t^2 + s^2 + z^2)^{3/2}} .$$

By comparison with J_1 this may now be expressed as

$$J_2 = \alpha_1 \left\{ \left[\tan^{-1}\frac{s_n}{z} - \tan^{-1}\frac{s_{n+1}}{z}\right] + \right.$$
$$\left. + \left[\tan^{-1}\frac{x_i s_{n+1}}{z\sqrt{z^2 + x_i^2 + s_{n+1}^2}} - \tan^{-1}\frac{x_i s_n}{z\sqrt{z^2 + x_i^2 + s_n^2}}\right]\right\} +$$
$$+ \alpha_2 \left\{ \frac{z}{2}\left[\ln(s_n^2 + z^2) - \ln(s_{n+1}^2 + z^2)\right] - \right.$$
$$\left. - z\left[\coth^{-1}\frac{\sqrt{z^2 + x_i^2 + s_{n+1}^2}}{x_i} - \coth^{-1}\frac{\sqrt{z^2 + x_i^2 + s_n^2}}{x_i}\right]\right\} .$$

The contribution of this to $J_w^{1,n}(x_i,y_j)$ is

$$\lim_{z\to 0} \frac{\partial J_2}{\partial z} = [\alpha_1, \alpha_2] \cdot \begin{Bmatrix} b_{1,2} \\ b_{2,2} \end{Bmatrix} , \tag{B-5}$$

where

$$b_{1,2} = \left(\frac{1}{s_{n+1}} - \frac{1}{s_n}\right) - \frac{1}{x_i}\left(\frac{S_{n+1}^{i,j}}{s_{n+1}} - \frac{S_n^{i,j}}{s_n}\right) ,$$

$$b_{2,2} = \ln\left(\frac{s_n}{s_{n+1}}\right) - \left[\coth^{-1}\left(\frac{S_{n+1}^{i,j}}{x_i}\right) - \coth^{-1}\left(\frac{S_n^{i,j}}{x_i}\right)\right] . \tag{B-6}$$

In the third term in (B-1), the integral over t has an improper upper limit and consequently changing the order of integration would not be desirable. To avoid this difficulty, we write this term as follows,

$$J_3 = -\frac{1}{\pi} \sum_{k=i}^{\infty} \int_{x_k}^{x_{k+1}} dt \int_{s_n}^{s_{n+1}} (\alpha_1+\alpha_2 s) ds \int_{-\pi/2}^{\pi/2} \sin\theta \, d\theta \; .$$

$$\times \int_0^{\infty} e^{-\rho} \cdot R_e \left[\frac{i}{(\chi + i\omega)^2} - \frac{i}{(\chi - i\omega)^2} \right] d\rho \; .$$

Now this integral is in a proper form and is similar to the integral I_3 in Appendix C. Therefore the contribution of this to $J_{\omega}^{m,n}(x_i, y_j)$ can be expressed in the form

$$\lim_{z \to 0} \frac{\partial J_3}{\partial z} = [\, \alpha_1, \alpha_2 \,] \cdot \begin{pmatrix} b_{1,3} \\ b_{2,3} \end{pmatrix} \quad , \qquad \text{(B-7)}$$

where

$$b_{1,3} = \sum_{k=i}^{k} a_{1,3} \quad ,$$

$$b_{2,3} = \sum_{k=i}^{k} a_{2,3} \quad . \qquad \text{(B-8)}$$

For numerical computations, the upper limit of summation has changed to a finite value k that is found to achieve the desired numerical accuracy.

Finally we will evaluate the contribution of the last term in (B-1). The value of this integral will depend upon the sign of $\bar{\omega}$. Since

$$\bar{\omega} = (x_i - \xi) \cos\theta + z \sin\theta$$

where

$$\xi < 0 \quad \text{and} \quad x_i > 0 \quad ,$$

therefore the sign of $\bar{\omega}$ will depend upon z sinθ. In the limit as $z \to 0$, $\bar{\omega} > 0$ and consequently $H(-\bar{\omega}) = \delta(-\bar{\omega}) = 0$. Hence it follows that the contribution of this integral to $J_w^{1,n}(x_i,y_j)$ vanishes. This result is expected since this integral represents the contribution due to the waves generated behind the doublets at $\xi = 0$. We now have the following formula for $J_w^{1,n}(x_i,y_j)$

$$J_w^{1,n}(x_i,y_j) = [\beta_1,\beta_2] \cdot \begin{pmatrix} b_{1,1} \\ b_{2,1} \end{pmatrix} + [\alpha_1,\alpha_2] \cdot \begin{pmatrix} b_{1,2}+b_{1,3} \\ b_{2,2}+b_{2,3} \end{pmatrix} \tag{B-9}$$

APPENDIX C

The $J_s^{m,n}(x_i,y_j)$ Integral

This integral is defined by equation (III-14). Applying the formula for integration by parts to the third term of this integral, we obtain

$$- \gamma_o \sec^2\theta \; Re \; i \int_o^\infty e^{-\rho} \, d\left[\frac{1}{x + i\omega} - \frac{1}{x - i\omega}\right]$$

$$= \gamma_o \sec^2\theta \left\{\frac{2\omega}{s^2 + \omega^2} - \int_o^\infty e^{-\rho} \, \frac{2\omega}{x^2 + \omega^2} \, d\rho\right\}$$

$$= \gamma_o \sec^2\theta \int_o^\infty e^{-\rho} \left[\frac{2\omega}{s^2 + \omega^2} - \frac{2\omega}{x^2 + \omega^2}\right] d\rho \; .$$

After changing the order of integration of the last three integrals in $J_s^{m,n}(x_i,y_j)$, it may be written as

$$J_s^{m,n}(x_i,y_i) = \lim_{z \to o} \frac{\partial}{\partial z} \left\{ z \int_{q_n}^{q_{n+1}} dq \int_{t_m}^{t_{m+1}} \frac{(\beta_1 + \beta_2 q) + (\beta_3 + \beta_4 q)t}{(t^2 + q^2 + z^2)^{3/2}} \, dt \; + \right.$$

$$+ \; z \int_{s_n}^{s_{n+1}} ds \int_{t_m}^{t_{m+1}} \frac{(\alpha_1 + \alpha_2 \, s) + (\alpha_3 + \alpha_4 \, s)t}{(t^2 + s^2 + z^2)^{3/2}} \, dt \; +$$

$$+ \frac{\gamma_o}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \sec^2\theta \, d\theta \int_o^\infty e^{-\rho} d\rho \int_{s_n}^{s_{n+1}} ds \int_{\omega_m}^{\omega_{m+1}} \Big[(\alpha_1 + \alpha_2 s) -$$

$$- (\alpha_3 + \alpha_4 s) z \tan\theta + (\alpha_3 + \alpha_4 s) \sec\theta \cdot \omega \Big] \cdot \Big[\frac{2\omega}{s^2 + \omega^2} -$$

$$- \frac{2\omega}{x^2 + \omega^2} \Big] d\omega + 4\gamma_o^2 \int_1^\infty \sin (\gamma_o \lambda \sqrt{\lambda^2 - 1} \cdot z) \cdot$$

$$\cdot \lambda^2 d\lambda \int_{s_n}^{s_{n+1}} e^{\gamma_o \lambda^2 s} ds \int_{t_m}^{t_{m+1}} H(\overline{\omega}) \cdot \sin(\gamma_o \lambda t) \cdot$$

$$\cdot [(\alpha_1 + \alpha_2 s) + (\alpha_3 + \alpha_4 s) \cdot t] \, dt +$$

$$+ 2\gamma_o \int_1^\infty d\lambda \int_{s_n}^{s_{n+1}} e^{\gamma_o \lambda^2 s} ds \int_{t_m}^{t_{m+1}} \delta(\overline{\omega}) \sin(2\gamma_o \lambda t) \cdot$$

$$\cdot [(\alpha_1 + \alpha_2 s) + (\alpha_3 + \alpha_4 s) \cdot t] \, dt \qquad \text{(C-1)}$$

Each of the above integrals will be evaluated separately; also the limit of its derivative with respect to z as $z \to o$ will be calculated. Consider the first integral, which can be written as

$$I_1 = z \int_{q_n}^{q_{n+1}} dq \int_{t_m}^{t_{m+1}} \frac{(\beta_1 + \beta_2 q) + (\beta_3 + \beta_4 q) t}{(t^2 + q^2 + z^2)^{3/2}} dt \, .$$

Integrating once over t,

$$I_1 = z \left\{ \int_{q_n}^{q_{n+1}} \frac{(\beta_1 + \beta_2 q)}{q^2 + z^2} \cdot \left[\frac{t_{m+1}}{\sqrt{q^2 + t_{m+1}^2 + z^2}} - \frac{t_m}{\sqrt{q^2 + t_m^2 + z^2}} \right] dq - \int_{q_n}^{q_{n+1}} (\beta_3 + \beta_4 q) \cdot \left[\frac{1}{\sqrt{q^2 + t_{m+1}^2 + z^2}} - \frac{1}{\sqrt{q^2 + t_m^2 + z^2}} \right] dq \right\}.$$

A similar integral to the first one above has been performed in Appendix B; therefore

$$I_1 = \beta_1 \left\{ \left[\tan^{-1} \left(\frac{t_{m+1} q_{n+1}}{z \cdot T_{m+1,n+1}} \right) - \tan^{-1} \left(\frac{t_{m+1} q_n}{z \cdot T_{m+1,n}} \right) \right] \cdot \left[\tan^{-1} \left(\frac{t_m q_{n+1}}{z \cdot T_{m,n+1}} \right) - \tan^{-1} \left(\frac{t_m q_n}{z \cdot T_{m,n}} \right) \right] \right\}$$

$$+ \; -\beta_2 \cdot z \left\{ \left[\coth^{-1}\left(\frac{T_{m+1,n+1}}{t_{m+1}}\right) - \coth^{-1}\left(\frac{T_{m+1,n}}{t_{m+1}}\right) \right] - \right.$$

$$\left. - \left[\coth^{-1}\left(\frac{T_{m,n+1}}{t_m}\right) - \coth^{-1}\left(\frac{T_{m,n}}{t_m}\right) \right] \right\} \cdot$$

$$+ \; -\beta_3 \cdot z \left\{ \left[\coth^{-1}\left(\frac{T_{m+1,n+1}}{q_{n+1}}\right) - \coth^{-1}\left(\frac{T_{m+1,n}}{q_n}\right) \right] - \right.$$

$$\left. - \left[\coth^{-1}\left(\frac{T_{m,n+1}}{q_{n+1}}\right) - \coth^{-1}\left(\frac{T_{m,n}}{q_n}\right) \right] \right\} \cdot$$

$$+ \; -\beta_4 \cdot z \left\{ (T_{m+1,n+1} - T_{m+1,n}) - (T_{m,n+1} - T_{m,n}) \right\} ,$$

where

$$T = \sqrt{z^2 + t^2 + q^2} \quad .$$

Differentiating this equation with respect to z and taking the limit as $z \to 0$ we obtain

$$\lim_{z \to 0} \frac{\partial I_1}{\partial z} = \beta_1 \left\{ \frac{1}{t_{m+1}} \left[\frac{Q^{i,j}_{m+1,n}}{q_n} - \frac{Q^{i,j}_{m+1,n+1}}{q_{n+1}} \right] - \right.$$

$$- \frac{1}{t_m} \cdot \left[\frac{Q^{i,j}_{m,n}}{q_n} - \frac{Q^{i,j}_{m,n+1}}{q_{n+1}} \right] \Bigg\}$$

$$+ \ \beta_2 \left\{ \left[\coth^{-1}\left(\frac{Q^{i,j}_{m+1,n}}{t_{m+1}} \right) - \coth^{-1}\left(\frac{Q^{i,j}_{m+1,n+1}}{t_{m+1}} \right) \right] - \right.$$

$$\left. - \left[\coth^{-1}\left(\frac{Q^{i,j}_{m,n}}{t_m} \right) - \coth^{-1}\left(\frac{Q^{i,j}_{m,n+1}}{t_m} \right) \right] \right\}$$

$$+ \ \beta_3 \left\{ \left[\coth^{-1}\left(\frac{Q^{i,j}_{m+1,n}}{q_n} \right) - \coth^{-1}\left(\frac{Q^{i,j}_{m+1,n+1}}{q_{n+1}} \right) \right] - \right.$$

$$\left. - \left[\coth^{-1}\left(\frac{Q^{i,j}_{m,n}}{q_n} \right) - \coth^{-1}\left(\frac{Q^{i,j}_{m,n+1}}{q_{n+1}} \right) \right] \right\}$$

$$+ \ \beta_4 \left\{ (Q^{i,j}_{m+1,n} - Q^{i,j}_{m+1,n+1}) - (Q^{i,j}_{m,n} - Q^{i,j}_{m,n+1}) \right\} , \qquad \text{(C-2)}$$

where

$$Q^{i,j} = \sqrt{t^2 + q^2} \ . \qquad \text{(C-3)}$$

Then the contribution of I_I to $J^{m,n}_s(x_i,y_i)$ can be expressed in the following form

$$\lim_{z \to 0} \frac{\partial I_1}{\partial z} = \left[\beta_1, \beta_2, \beta_3, \beta_4 \right] \cdot \begin{Bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ a_{4,1} \end{Bmatrix} \qquad \text{(C-4)}$$

Consider the second integral in (C-1). This can be written as

$$I_2 = z \int_{s_n}^{s_{n+1}} ds \int_{t_m}^{t_{m+1}} \frac{(\alpha_1 + \alpha_2 s) + (\alpha_3 + \alpha_4 s)\cdot t}{(t^2 + s^2 + z^2)^{3/2}} \, dt$$

By comparison with I_1, the contribution of this integral to $J_s^{m,n}(x_i, y_j)$ can be expressed in the form

$$\lim_{z \to 0} \frac{\partial I_2}{\partial z} = \left[\alpha_1, \alpha_2, \alpha_3, \alpha_4 \right] \cdot \begin{Bmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \\ a_{4,2} \end{Bmatrix} , \qquad \text{(C-5)}$$

where

$$a_{1,\binom{1}{2}} = \frac{1}{t_{m+1}} \left[\binom{Q}{S}^{i,j}_{m+1,n} \Big/ \binom{q}{s}_n - \binom{Q}{S}^{i,j}_{m+1,n+1} \Big/ \binom{q}{s}_{n+1} \right] -$$

$$- \frac{1}{t_m} \left[\left(\frac{Q}{S}\right)^{i,j}_{m,n} / \left(\frac{q}{s}\right)_n - \left(\frac{Q}{S}\right)^{i,j}_{m,n+1} / \left(\frac{q}{s}\right)_{n+1} \right],$$

$$a_{2,\left(\frac{1}{2}\right)} = \left[\coth^{-1} \left(\frac{\left(\frac{Q}{S}\right)^{i,j}_{m+1,n}}{t_{m+1}} \right) - \coth^{-1} \left(\frac{\left(\frac{Q}{S}\right)^{i,j}_{m+1,n+1}}{t_{m+1}} \right) \right] -$$

$$- \left[\coth^{-1} \left(\frac{\left(\frac{Q}{S}\right)^{i,j}_{m,n}}{t_m} \right) - \coth^{-1} \left(\frac{\left(\frac{Q}{S}\right)^{i,j}_{m,n+1}}{t_m} \right) \right],$$

$$a_{3,\left(\frac{1}{2}\right)} = \left\{ \coth^{-1} \left[\left(\frac{Q}{S}\right)^{i,j}_{m+1,n} / \left(\frac{q}{s}\right)_n \right] - \coth^{-1} \left[\left(\frac{Q}{S}\right)^{i,j}_{m+1,n+1} / \left(\frac{q}{s}\right)_{n+1} \right] \right\} -$$

$$- \coth^{-1} \left[\left(\frac{Q}{S}\right)^{i,j}_{m,n} / \left(\frac{q}{s}\right)_n \right] + \coth^{-1} \left[\left(\frac{Q}{S}\right)^{i,j}_{m,n+1} / \left(\frac{q}{s}\right)_{n+1} \right] \Bigg\},$$

$$a_{4,\left(\frac{1}{2}\right)} = \left[\left(\frac{Q}{S}\right)^{i,j}_{m+1,n} - \left(\frac{Q}{S}\right)^{i,j}_{m+1,n+1} \right] - \left[\left(\frac{Q}{S}\right)^{i,j}_{m,n} - \left(\frac{Q}{S}\right)^{i,j}_{m,n+1} \right],$$

(C-6)

where

$$S = \sqrt{t^2 + s^2} .$$

Consider the third integral in (C-1), which will be written as

$$I_3 = \frac{\gamma_o}{\pi} \int_{-\pi/2}^{\pi/2} \tan\theta \sec^2\theta \, d\theta \int_0^{\infty} e^{-\rho} \, d\rho \int_{s_n}^{s_{n+1}} ds \int_{\omega_m}^{\omega_{m+1}} \Big| [(\alpha_1+\alpha_2 s)-(\alpha_3+\alpha_4 s)\cdot Z \tan\theta]+$$

$$+ [(\alpha_3+\alpha_4 s)\omega \sec\theta]) \Big| \, d\left(\ln \frac{s^2+\omega^2}{x^2+\omega^2}\right) .$$

(C-7)

By integrating the last term over ω, we have

$$\Big[(\alpha_1+\alpha_2 s)-(\alpha_3+\alpha_4 s)\cdot Z \tan\theta\Big]\cdot\left[\ln\left(\frac{s^2+\omega_{m+1}^2}{x^2+\omega_{m+1}^2}\right)-\ln\left(\frac{s^2+\omega_m^2}{x^2+\omega_m^2}\right)\right]+$$

$$+(\alpha_3+\alpha_4 s)\cdot\sec\theta\cdot\left\{\left[\omega_{m+1} \ln\left(\frac{s^2+\omega_{m+1}^2}{x^2+\omega_{m+1}^2}\right) - \omega_m \ln\left(\frac{s^2+\omega_m^2}{x^2+\omega_m^2}\right)\right]\right.$$

$$\left.-\int_{\omega_m}^{\omega_{m+1}} \ln \frac{s^2+\omega^2}{x^2+\omega^2} \, d\omega\right\}$$

$$= \Big[(\alpha_1+\alpha_2 s)-(\alpha_3+\alpha_4 s)\cdot Z \tan\theta\Big] \cdot \left[\ln\left(\frac{s^2+\omega_{m+1}^2}{x^2+\omega_{m+1}^2}\right) -\right.$$

$$\left.-\ln\left(\frac{s^2+\omega_m^2}{x^2+\omega_m^2}\right)\right] -2(\alpha_3+\alpha_4 s) \sec\theta \cdot \left\{ s\left[\tan^{-1}\left(\frac{\omega_{m+1}}{s}\right) -\right.\right.$$

$$\left.\left.-\tan^{-1}\left(\frac{\omega_m}{s}\right)\right] - x\left[\tan^{-1}\left(\frac{\omega_{m+1}}{x}\right) - \tan^{-1}\left(\frac{\omega_m}{x}\right)\right]\right\} .$$

Since this equation is in a proper form, then it is possible now to take the limit of the derivative with respect to Z as $Z\to o$ before integrating over S. Than it becomes

$$(\alpha_1+\alpha_2 s)\cdot 2\sin\theta\cos\theta\cdot\left\{t_{m+1}\left[\frac{1}{s^2+t_{m+1}^2\,.\cos^2\theta}-\frac{1}{x^2+t_{m+1}^2\cos^2\theta}\right]-\right.$$

$$\left.-t_m\left[\frac{1}{s^2+t_m^2\cos^2\theta}-\frac{1}{x^2+t_m^2\cos^2\theta}\right]\right\}-(\alpha_3+\alpha_4 s)\cdot\tan\theta\ .$$

$$\left[\ln\left(\frac{s^2+t_{m+1}^2\cos^2\theta}{x^2+t_{m+1}^2\cos^2\theta}\right)-\ln\left(\frac{s^2+t_m^2\cos^2\theta}{x^2+t_m^2\cos^2\theta}\right)\right]+2\left[s^2\left(\frac{1}{s^2+t_{m+1}^2\cos^2\theta}-\right.\right.$$

$$\left.\left.-\frac{1}{s^2+t_m^2\cos^2\theta}\right)-x^2\left(\frac{1}{x^2+t_{m+1}^2\cos^2\theta}\right)-\left(\frac{1}{x^2+t_m^2\cos^2\theta}\right)\right]\ .$$

Integrating now over s, we obtain

$$\cdot\alpha_1\cdot 2\sin\theta\left\{\left[\tan^{-1}\left(\frac{2\Delta_\eta\, t_{m+1}\cos\theta}{t_{m+1}^2\cos^2+s_n\cdot s_{n+1}}\right)-\tan^{-1}\left(\frac{2\Delta_\eta\, t_{m+1}\cos}{t_{m+1}^2\cos^2\theta+x_n\cdot x_{n+1}}\right)\right]-\right.$$

$$\left.\left[\tan^{-1}\left(\frac{2\Delta_\eta\, t_m\cos\theta}{t_m^2\cos^2\theta+s_n\cdot s_{n+1}}\right)-\tan^{-1}\left(\frac{2\Delta_\eta\, t_m\cos\theta}{t_m^2\cos^2\theta+x_n\cdot x_{n+1}}\right)\right]\right\}$$

$$+\alpha_2\cdot\sin\theta\cos\theta\left\{\left[t_{m+1}\ln\left(\frac{\Gamma_{m+1,n+1}^{i,j}}{\Gamma_{m+1,n}^{i,j}}\right)-t_m\ln\left(\frac{\Gamma_{m,n+1}^{i,j}}{\Gamma_{m,n}^{i,j}}\right)\right]-\left[t_{m+1}\ln\left(\frac{\Omega_{m+1,n+1}^{i,j}}{\Omega_{m+1,n}^{i,j}}\right)-t_m\right.\right.$$

$$\left.\left.\ln\left(\frac{\Omega_{m,n+1}^{i,j}}{\Omega_{m,n}^{i,j}}\right)\right]+2\,\frac{\rho\cos^2}{\gamma_o}\cdot\sec\theta\left[\tan^{-1}\left(\frac{2\Delta_\eta\, t_{m+1}\cos\theta}{t_{m+1}^2\cos^2\theta+x_n\cdot x_{n+1}}\right)-\tan^{-1}\left(\frac{2\Delta_\eta\, t_m\cos\theta}{t_m^2\cos^2\theta+x_n\cdot x_{n+1}}\right)\right]\right\}$$

$$-\alpha_3 \cdot \tan\theta \left\{ \left[s_{n+1} \ln\left(\Gamma^{i,j}_{m+1,n+1} / \Gamma^{i,j}_{m,n+1}\right) - s_n \ln\left(\Gamma^{i,j}_{m+1,n} / \Gamma^{i,j}_{m,n}\right) \right] - \right.$$

$$\left. - \left[x_{n+1} \ln\left(\Omega^{i,j}_{m+1,n+1} / \Omega^{i,j}_{m,n+1}\right) - x_n \ln\left(\Omega^{i,j}_{m+1,n} / \Omega^{i,j}_{m,n}\right) \right] \right\}$$

$$- \frac{1}{2} \alpha_4 \tan\theta \left\{ \left[(t^2_{m+1} \cos^2\theta - s^2_{n+1}) \ln\left(\Omega_{m+1,n+1} / \Gamma_{m+1,n+1}\right) \right. \right.$$

$$- (t^2_{m+1} \cos^2\theta - s^2_n) \ln\left(\Omega_{m+1,n} / \Gamma_{m+1,n}\right)$$

$$- (t^2_m \cos^2\theta - s^2_{n+1}) \ln\left(\Omega_{m,n+1} / \Gamma_{m,n+1}\right)$$

$$\left. + (t^2_m \cos^2\theta - s^2_n) \ln\left(\Omega_{m,n} / \Gamma_{m,n}\right) \right]$$

$$\left. + \left(\frac{\rho \cos^2\theta}{\gamma_0}\right)^2 \left[\ln\left(\Omega_{m+1,n+1} / r_{m+1,n}\right) - \ln\left(\Omega_{m,n+1} / r_{m,n}\right) \right] \right\}$$

where

$$\Gamma^{i,j} = s^2 + t^2 \cos^2\theta ,$$

$$\Omega^{i,j} = x^2 + t^2 \cos^2\theta .$$

$$\text{(C-9)}$$

The contribution of I_3 to $J^{m,n}_s(x_i, y_j)$ can then be expressed as

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4] \cdot \begin{Bmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \\ a_{4,3} \end{Bmatrix} , \qquad \text{(C-10)}$$

where from (C-6) and (C-7)

$$a_{1,3} = \frac{4\gamma_o}{\pi} \int_o^\infty e^{-\rho} d\rho \int_1^\infty \sqrt{\lambda^2-1} \left\{ \left[\tan^{-1} \left(\frac{2\Delta_\eta \, t_{m+1} \, \lambda}{t_{m+1}^2 + S_n \, S_{n+1} \, \lambda^2} \right) - \right.\right.$$

$$\left. \tan^{-1} \left(\frac{2\Delta_\eta \, t_{m+1} \, \lambda}{t_{m+1}^2 + x_n \, x_{n+1} \, \lambda^2} \right) \right] - \left[\tan^{-1} \left(\frac{2\Delta_\eta \, t_m \, \lambda}{t_m^2 + S_n \, S_{n+1} \, \lambda^2} \right) - \right.$$

$$\left.\left. -\tan^{-1} \left(\frac{2\Delta_\eta \, t_m \, \lambda}{t_m^2 + x_n \, x_{n+1} \, \beta^2} \right) \right] \right\} d\lambda$$

$$a_{2,3} = \frac{2\gamma_o}{\pi} \int_o^\infty e^{-\rho} d\rho \left\{ \int_1^\infty \frac{\sqrt{\lambda^{2-1}}}{\lambda} \{ t_{m+1} \; [\ln (\Omega_{m+1,n}/\Gamma_{m+1,n}) - \right.$$

$$- \ln (\Omega_{m+1,n+1}/\Gamma_{m+1,n+1})] - t_m \; [\ln (\Omega_{m,n}/\Gamma_{m,n}) -$$

$$- \ln (\Omega_{m,n+1}/\Gamma_{m,n+1})] \} \, d\lambda$$

$$\left. + 2(\frac{\rho}{\gamma_o}) \int_1^\infty \frac{\sqrt{\lambda^{2-1}}}{\lambda^2} \left| \tan^{-1} \left(\frac{2\Delta_\eta \, t_{m+1} \, \lambda}{t_{m+1}^2 + x_n \, x_{n+1} \, \lambda^2} \right) - \tan^{-1} \left(\frac{2\Delta_\eta \, t_m \, \lambda}{t_m + x_n \, x_{n+1} \, \lambda^2} \right) \right| d\lambda \right\}$$

$$a_{3,3} = \frac{\gamma_o}{\pi} \int_o^\infty e^{-\rho} \, d\rho \left\{ \int_1^\infty \lambda \sqrt{\lambda^{2-1}} \; \{ S_{n+1} \; [\ln (\Omega_{m+1,n+1}/\Gamma_{m+1,n+1}) \right.$$

$$- \ln (\Omega_{m,n+1}/\Gamma_{m,n+1})] - S_n \; [\ln (\Omega_{m+1}/\Gamma_{m+1,n})$$

$$- \ln (\Omega_{m,n}/\Gamma_{m,n})] \} \, d\lambda \; +$$

$$\left. + (\frac{\rho}{\gamma_o}) \int_1^\infty \frac{\sqrt{\lambda^{2-1}}}{\lambda} \; \{ \ln (\Omega_{m+1,n+1}/\Omega_{m+1,n}) - \ln (\Omega_{m,n+1}/\Omega_{m,n}) \} \, d\lambda \right\}$$

$$a_{4,3} = \frac{\gamma_o}{\pi} \int_0^\infty e^{-\rho}\, d\rho \left\{ \int_1^\infty \lambda\sqrt{\lambda^{2-1}} \; \{S^2_{n+1} \; [\ln\,(\Omega_{m+1,n+1}/\Gamma_{m+1,n+1})\right.$$

$$- \ln\,(\Omega_{m,n+1}/\Gamma_{m,n+1})] - S^2_n \; [\ln\,(\Omega_{m+1,n}/\Gamma_{m+1,n}) - \ln\,(\Omega_{m,n}/\Gamma_{m,n})]\}\, d\lambda$$

$$- \int_1^\infty \frac{\sqrt{\lambda^{2-1}}}{\lambda} \; \{t^2_{m+1} \; [\ln\,(\Omega_{m+1,n+1}/\Gamma_{m+1,n+1}) - \ln\,(\Omega_{m+1,n}/\Gamma_{m+1,n})]$$

$$- t^2_m \; [\ln\,(\Omega_{m,n+1}/\Gamma_{m,n+1}) - \ln\,(\Omega_{m,n}/\Gamma_{m,n})]\} \; d\lambda$$

$$\left. - \left(\frac{\rho}{\gamma_o}\right)^2 \int_1^\infty \frac{\sqrt{\lambda^{2-1}}}{\lambda} \; \{\ln\,(\Omega_{m+1,n+1}/\Omega_{m+1,n}) - \ln\,(\Omega_{m,n+1}/\Omega_{m,n})\} \; d\lambda \right\}$$

where we have made the substitution $\lambda = \sec\theta$ in the above integrals.

Consider next the fourth term in (C-1),

$$I_4 = 4\,\gamma_o^2 \int_1^\infty \sin\,(\gamma_o\lambda\,\sqrt{\lambda^{2-1}}\cdot z)\cdot\lambda^2 d\lambda \int_{s_n}^{s_{n+1}} e^{\gamma_o\lambda^2} ds \int_{t_m}^{t_{m+1}} H(\bar{\omega})\,\sin(\gamma_o\lambda t)\;.$$

$$[(\alpha_1 + \alpha_2\, s) + (\alpha_3 + \alpha_4\, s)\cdot t]\; dt\;.$$

It is possible here to take the limit of the derivative with respect to z as $z \to 0$ before integrating over t and s. Then

$$\lim_{z\to 0} \frac{\partial I_4}{\partial z} = 4\,\gamma_o^3 \int_1^\infty \lambda^3 \sqrt{\lambda^2-1}\; d\lambda \int_{s_n}^{s_{n+1}} e^{\gamma_o \lambda^2} ds \int_{t_m}^{t_{m+1}} H(-t)\, \sin\,(\gamma_o \lambda t)$$

$$[(\alpha_1 + \alpha_2 s) + (\alpha_3 + \alpha_4 s)\cdot t]\; dt \quad .$$

Integrating over t , then

$$\lim_{z\to 0} \frac{\partial I_4}{\partial z} = -4\,\gamma_o^2 \int_1^\infty \lambda^2 \sqrt{\lambda^2-1}\; d\lambda \int_{s_n}^{s_{n+1}} e^{\gamma_o \lambda^2 s}\, \{(\alpha_1+\alpha_2 s)\cdot g_1 + (\alpha_3+\alpha_4 s)\cdot g_2\}\, ds \; ,$$

where

$$g_1 = H\,(-t_{m+1}) \cos(\gamma_o \lambda\; t_{m+1}) - H(-t_m) \cos(\gamma_o \lambda\; t_m) \quad ,$$

$$g_2 = H\,(-t_{m+1})\cdot\, [t_{m+1}\cdot \cos(\gamma_o \lambda\; t_{m+1}) - \frac{1}{\gamma_o\, \lambda} \sin\,(\gamma_o \lambda\; t_{m+1})] -$$

$$- H\,(-t_m)\cdot\, [t_m \cdot \cos(\gamma_o \lambda\; t_m) - \frac{1}{\gamma_o \lambda} \sin\,(\gamma_o \lambda\; t_m)] \; . \tag{C-12}$$

Integrating over s , we obtain

$$\lim_{z\to 0} \frac{\partial I_4}{\partial} = -4\,\gamma_o \int_1^\infty \sqrt{\lambda^2-1}\; \{(\alpha_1\, g_1 + \alpha_3\, g_2)\cdot g_3 + (\alpha_2\, g_1 + \alpha_4\, g_2)\cdot g_4\}\, d\lambda \quad ,$$

where

$$g_3 = e^{\gamma_o \lambda^2 s_{n+1}} - e^{\gamma_o \lambda^2 s_n} \; ,$$

$$g_4 = e^{\gamma_o \lambda^2 s_{n+1}} (s_{n+1} - \frac{1}{\gamma_o \lambda^2}) - e^{\gamma_o \lambda^2 s_n}\, (s_n - \frac{1}{\gamma_o \lambda^2}) \; . \tag{C-13}$$

The contribution of this integral to $J_s^{m,n}(x_i, y_j)$ can be expressed as

$$[\alpha_k] \cdot \{a_{k,4}\} \qquad \text{and} \quad k = 1,2,3,4 \quad , \tag{C-14}$$

where

$$a_{1,4} = -4\gamma_o \int_1^\infty \sqrt{\lambda^2-1} \cdot g_1 \cdot g_2 \cdot d\lambda \quad ,$$

$$a_{2,4} = -4\gamma_o \int_1^\infty \sqrt{\lambda^2-1} \cdot g_1 \cdot g_4 \cdot d\lambda \quad ,$$

$$a_{3,4} = -4\gamma_o \int_1^\infty \sqrt{\lambda^2-1} \cdot g_2 \cdot g_3 \cdot d\lambda \quad ,$$

$$a_{4,4} = -4\gamma_o \int_1^\infty \sqrt{\lambda^2-1} \cdot g_2 \cdot g_4 \cdot d\lambda \quad . \tag{C-15}$$

Finally the last integral in (C-1) can be written as

$$I_5 = 2\gamma_o \int_1^\infty d\lambda \int_{s_n}^{s_{n+1}} e^{\gamma_o\lambda^2 s}\, ds \int_{t_m}^{t_{m+1}} \delta(\bar{\omega}) \cdot$$

$$\cdot [(\alpha_1 + \alpha_2 s) + (\alpha_3 + \alpha_4 s) \cdot t] \sin(2\gamma_o\lambda t)\, dt \tag{C-16}$$

integrating over t we obtain

$$I_5 = -2\gamma_o \int_1^\infty \lambda\, d\lambda \int_{s_n}^{s_{n+1}} e^{\gamma_o\lambda^2 s} [(\alpha_1 + \alpha_2 s) - (\alpha_3 + \alpha_4 s) \sqrt{\lambda^2-1} \cdot z]$$

$$\sin(2\gamma_o\lambda\sqrt{\lambda^2-1} \cdot z)\, ds$$

$$\text{for } \xi_m < x_i + z\sqrt{\lambda^2-1} < \xi_{m+1}$$

$$= 0 \qquad\qquad \text{otherwise.}$$

Differentiating with regard to z and taking the limit as $z \to 0$,

we obtain

$$\lim_{z\to 0} \frac{\partial I_5}{\partial z} = -4\gamma_o^2 \int_1^\infty \lambda^2 \sqrt{\lambda^2-1}\, d\lambda \int_{s_n}^{s_{n+1}} (\alpha_1 + \alpha_2 s)\, e^{\gamma_o \lambda^2 s}\, ds$$

$$= -4\gamma_o \int_1^\infty \sqrt{\lambda^2-1}\, [\alpha_1 \cdot g_3 + \alpha_2 \cdot g_4]\, d\lambda \quad , \quad \xi_m < x_i < \xi_{m+1} \, .$$

The contribution of this integral to $J_s^{m,n}(x_i,y_j)$ can be written as

$$[\alpha_k] \cdot \{a_{k,5}\} \qquad \text{and } k = 1,2,3,4 \quad , \quad \xi_m < x_i < \xi_{m+1} \, . \tag{C-17}$$

where

$$a_{1,5} = -4\gamma_o \int_1^\infty \sqrt{\lambda^2-1} \cdot g_3 \cdot d\lambda \quad ,$$

$$a_{2,5} = -4\gamma_o \int_1^\infty \sqrt{\lambda^2-1} \cdot g_4 \cdot d\lambda \quad ,$$

$$a_{3,5} = 0 \quad ,$$

$$a_{4,5} = 0 \quad .$$

We can now write $J_s^{m,n}(x_i,y_j)$ as

$$J_s^{m,n}(x_i,y_j) = [\beta_k] \cdot \{a_{k,1}\} + [\alpha_k] \cdot \{a_{k,2} + a_{k,3} + a_{k,4} + a_{k,5}\} \quad ,$$

$$\text{and } k = 1,2,3,4 \quad . \tag{C-18}$$

Appendix D

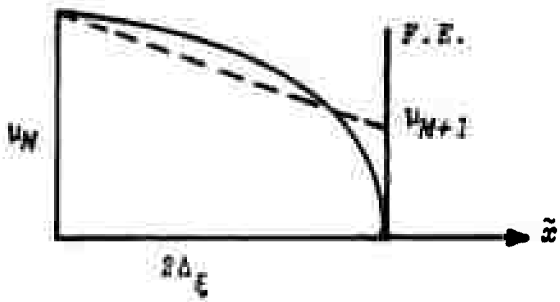
Numerical Analysis:

To determine the unknown values of μ numerically we have to compute the coefficients of the matrix given in (III-19) at a number of suitable control points (x_i, y_j) on s_o equal to the number of the unknown μ's . It is known from the potential theory of double distributions that the limit of the normal derivative of the potential exists on s_o under the condition that the moment μ have a continuous second derivative with respect to ξ and η in a neighborhood of the point (x_i, y_j) [we refer the reader to Kellogg, 1929, pp. 168] for more information. Accordingly, the control points should be taken inside the elements rather than at the nodal points for our analysis.

In order to be able to compute the values of μ near the forward end of the body accurately, a different perturbation scheme should be used to solve the nonlinear problem formulated in Chapter I near that edge. We thereby obtain a local solution that complements the solution we have, and will be matched with it in the next section [see Van Dyke, 1964, 4.9 for more discussion concerning this method of solution]. Instead, we will assume that the behavior of μ near to this

edge can be expressed by the following relation

$$\mu_n(x) = \mu_{M,n} \sqrt{1 - (\tilde{x}/2\Delta\xi)} \qquad \text{(1-D)}$$

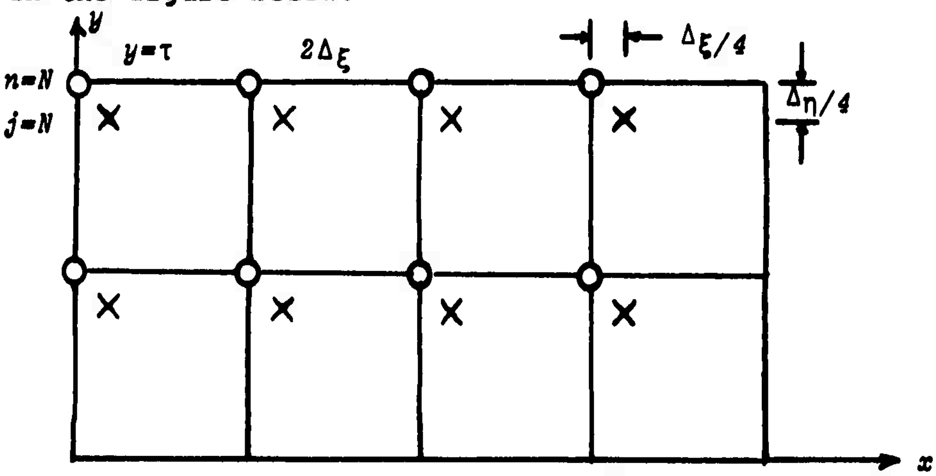


Furthermore, to simplify the numerical computations this relation is approximated by a straight line, which bounds the same area as that given by the above equation, namely,

$$\mu_{M+1} = \mu_M/3 \quad .$$

Obviously, there will be a discontinuity in the pressure jump at this forward end of the body. To obtain the forces, the integration of the pressure jump has to be extended to a point just off this end.

The control points (x_i, y_j) will be taken as shown in the figure below.



Now, to compute the coefficients of the matrix for these control points, we need some efficient means of evaluating the double intrgrals in (C-11), which we will denote by $D^{m,n}(a,b)$, and the integrals in (C-15), which we denote by $I^{m,n}(a,b)$.

The $D^{m,n}(a,b)$ Integrals:

These integrals can be written in the form,

$$D^{m,n}(a,b) = \int_0^\infty e^{-\rho}\, d\rho \int_1^\infty F(a,b;\rho,\lambda)\, d\lambda \quad .$$

We will approximate the integral with respect to ρ by using the integration formula for exponential integrals given by Abramowitz (1968, 25, 4.45). The result is the following:

$$D^{m,n}(a,b) = \sum_{i=1}^{n} w_i \cdot F_1(a,b;\rho_i) + R_n \quad , \qquad \text{(D-2)}$$

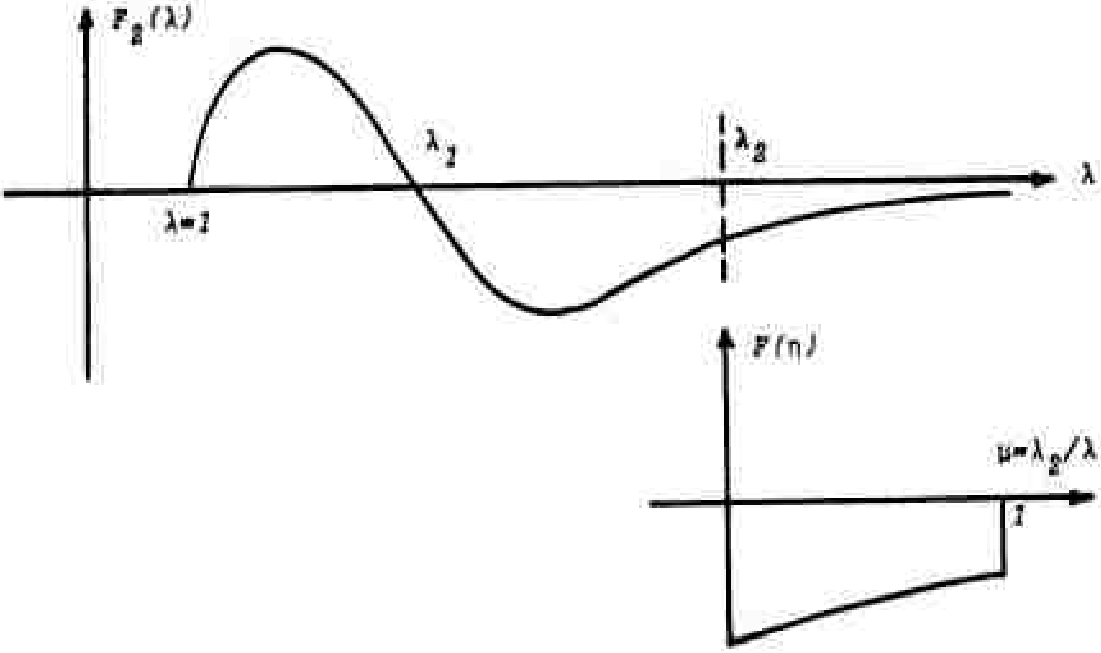
where ρ_i is the i^{th} zero of the Laguerre polynomial $L_n(\rho)$, w_i is the weight function given in tables by Abramowitz, and R_n is the truncation error,

$$R_n = \frac{(n!)^2}{(2n)!} F_1^{(2n)}(t,s;\xi) \quad , \quad (0 < \xi < \infty) \quad .$$

The function $F_1(t,s;\rho_i)$ is now defined by

$$F_1(a,b;\rho_i) = \int_1^\infty F(a,b;\rho_i,\lambda)\, d\lambda \quad .$$

The general behavior of the integrand $F(t,s;\rho_i,\lambda)$ is as shown in the figure below.



This integral is evaluated numerically as follows:

$$F_1(a,b;\rho_i) = \left[\int_1^{\lambda_1} + \int_{\lambda_1}^{\lambda_2}\right] F(a,b;\rho_i,\lambda)\, d\lambda \; +$$

$$+ \int_0^1 \frac{\lambda_2}{\mu^2} F(a,b;\rho_i,\mu)\, d\mu \quad , \qquad \text{(D-3)}$$

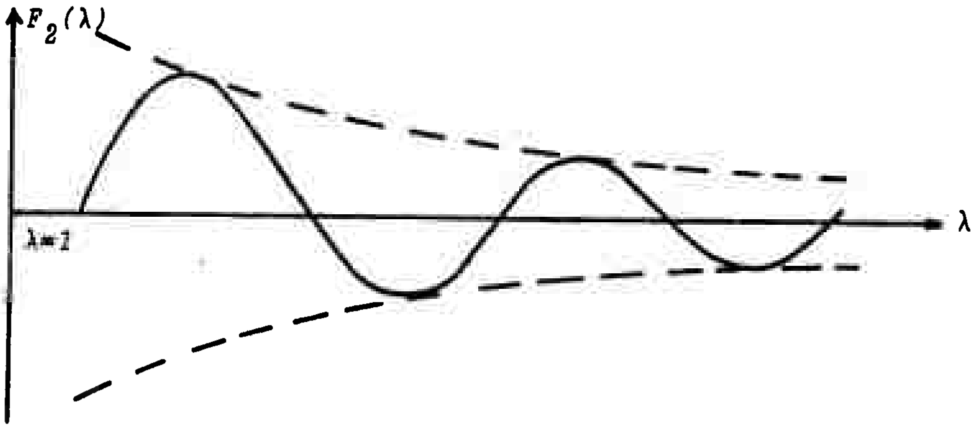
where λ_2 is the value of λ at which the function $F(\lambda)$ starts decreasing monotonically toward zero. Each of the above integrals in (D-3) is computed numerically for each value of ρ_i by using the trapezoidal rule.

<u>The $I^{m,n}(a,b)$ Integrals</u>:

These integrals can be written in the form

$$I^{m,n}(a,b) = \int_1^\infty F_2(a,b;\lambda)\,d\lambda \quad . \tag{D-4}$$

The general behavior of the integrand $F_2(a,b;\lambda)$ is as shown in the figure below



Numerically, the improper upper limit of integration will be replaced by a finite value λ_0 . A simple error expression $E(a,b;\lambda_0)$ can be derived as a function of the parameters a,b and λ_0 as follows:

$$E(a,b;\lambda_0) = \left| \int_{\lambda_0}^\infty F_2(a,b;\lambda)\,d\lambda \right| \quad .$$

The function $F_2(a,b;\lambda)$ in the above integral can be simplified considerably if we consider values of λ where $\lambda >> 1$. Consequently the above integral can be approximated analytically. This expression can be used to find values of λ_0 that achieve the desired numerical accuracy. Knowing λ_0 , the integral in (D-4) can be evaluated by the use of the trapezoidal rule.